Parametrized Families of Resolvent Compositions *

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Abstract This paper presents an in-depth analysis of a parametrized version of the resolvent composition, an operation that combines a set-valued operator and a linear operator. We provide new properties and examples, and show that resolvent compositions can be interpreted as parallel compositions of perturbed operators. Additionally, we establish new monotonicity results, even in cases when the initial operator is not monotone. Finally, we derive asymptotic results regarding operator convergence, specifically focusing on graph-convergence and the ρ-Hausdorff distance.

Keywords. Graph-convergence, monotone operator, parallel composition, resolvent composition, resolvent mixture, ρ-Hausdorff distance, set-convergence.

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§1. Introduction

Throughout, $\mathcal{H}$ is a real Hilbert space with power set $2^{\mathcal{H}}$, identity operator $\mathrm{Id}_{\mathcal{H}}$, scalar product $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$, and associated norm $\|\cdot\|_{\mathcal{H}}$. In addition, $\mathcal{G}$ is a real Hilbert space, the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$ is denoted by $\mathcal{B}(\mathcal{H},\mathcal{G})$, and $\mathcal{B}(\mathcal{H}) = B(\mathcal{H},\mathcal{H})$. Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$. The adjoint of L is denoted by L^*, and the parallel composition of a set-valued operator $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ by L^* is the operator from $\mathcal{H}$ to $2^{\mathcal{H}}$ given by

$$L^* \triangleright B = \left(L^* \circ B^{-1} \circ L\right)^{-1}. \tag{1.1}$$

We focus our attention on new methods to combine a set-valued operator with a linear operator, which have recently been introduced in [11], where they have been studied only for the case $\gamma = 1$.

Definition 1.1. Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, and let $\gamma \in \left]0,+\infty\right[$. The *resolvent composition* of B and L with parameter γ is the operator $L \overset{\gamma}{\diamond} B\colon \mathcal{H} \to 2^{\mathcal{H}}$ given by

$$L \overset{\gamma}{\diamond} B = L^* \triangleright \left(B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}\right) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} \tag{1.2}$$

and the *resolvent cocomposition* of B and L with parameter γ is the operator $L \overset{\gamma}{\blacklozenge} B\colon \mathcal{H} \to 2^{\mathcal{H}}$ given by $L \overset{\gamma}{\blacklozenge} B = (L \overset{1/\gamma}{\diamond} B^{-1})^{-1}$. Further, $L \diamond B = L \overset{1}{\diamond} B$ and $L \blacklozenge B = L \overset{1}{\blacklozenge} B$.

Resolvents of set-valued operators are essential in the numerical solution of monotone inclusion problems [12, 16, 18, 19, 21, 22]. A motivation for studying the resolvent compositions of Definition 1.1 stems from the fact that their resolvent can be computed explicitly, unlike those of the standard composite operators $L^* \circ B \circ L$ and $L^* \triangleright B$, for which the resolvent is typically intractable and requires dedicated numerical methods [1, 10, 15]. Resolvent compositions also show up in relaxations of inconsistent inclusion problems [8, 11]. For instance, these new composite operators can be utilized to model relaxations of convex feasibility and nonlinear reconstruction problems [17]. Furthermore, the resolvent composition of the subdifferential of a proper lower semicontinuous convex function is the subdifferential of a function called the *proximal composition* (see [8, 11, 14]), which has been used in image recovery and machine learning applications [13].

The goal of this paper is to present an in-depth analysis of the parametrized compositions of Definition 1.1. We provide various new properties and examples, as well as connections with connection with the parallel composition $L^* \triangleright B$ and the standard composition $L^* \circ B \circ L$. Additionally, we establish new monotonicity results, including the preservation of monotonicity, strongly monotonicity, and maximally monotonicity, and examine the Fitzpatrick function of the resolvent composition. Finally, we investigate the convergence of the operators $L \overset{\gamma}{\diamond} B$ and $L \overset{\gamma}{\blacklozenge} B$ as γ varies, by examining the graph-convergence and the ρ-Hausdorff distance convergence.

The remainder of the paper is organized as follows. In Section 2, we provide our notation and necessary mathematical background. In Section 3, we investigate new properties of the parametrized resolvent compositions and present several examples. Section 4 is devoted to study the monotonicity of resolvent compositions. Finally, Section 5 provides convergence results for parametrized resolvent compositions as the parameter varies.

§2. Notation and background

We first present our notation, which follows [7].

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The domain of A is $\operatorname{dom} A = \big\{x \in \mathcal{H} \mid Ax \neq \varnothing\big\}$, the range of A is $\operatorname{ran} A = \big\{x^* \in \mathcal{H} \mid (\exists\, x \in \mathcal{H})\, x^* \in Ax\big\}$, the graph of A is $\operatorname{gra} A = \big\{(x, x^*) \in \mathcal{H} \times \mathcal{H} \mid x^* \in Ax\big\}$, the zeros of A is $\operatorname{zer} A = \big\{x \in \mathcal{H} \mid 0 \in Ax\big\}$, the inverse of A is the set-valued operator A^{-1} with graph $\big\{(x^*, x) \in \mathcal{H} \times \mathcal{H} \mid x^* \in Ax\big\}$. The resolvent of A is

$$J_A = (\mathrm{Id}_{\mathcal{H}} + A)^{-1} \tag{2.1}$$

and the Yosida approximation of A of index $\gamma \in \left]0, +\infty\right[$ is

$$^{\gamma}\!A = \gamma^{-1}(\mathrm{Id}_{\mathcal{H}} - J_{\gamma A}) = (A^{-1} + \gamma \mathrm{Id}_{\mathcal{H}})^{-1}. \tag{2.2}$$

In particular, when $\gamma = 1$,

$$\mathrm{Id}_{\mathcal{H}} - J_A = J_{A^{-1}}. \tag{2.3}$$

The operator A is monotone if

$$\big(\forall (x_1, x_1^*) \in \operatorname{gra} A\big)\big(\forall (x_2, x_2^*) \in \operatorname{gra} A\big)\ \langle x_1 - x_2 \mid x_1^* - x_2^*\rangle_{\mathcal{H}} \geqslant 0, \tag{2.4}$$

α-strongly monotone for some $\alpha \in \left]0, +\infty\right[$ if $A - \alpha \mathrm{Id}_{\mathcal{H}}$ is monotone, and maximally monotone if it is monotone and there exists no monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gra} B$ properly contains $\operatorname{gra} A$.

Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. If $\operatorname{ran} L$ is closed, the generalized (or Moore–Penrose) inverse of L is denoted by $L^{\dagger}$. Further, L is an isometry if $L^* \circ L = \mathrm{Id}_{\mathcal{H}}$ and a coisometry if $L \circ L^* = \mathrm{Id}_{\mathcal{G}}$.

Let D be a nonempty subset of $\mathcal{H}$ and let $T\colon D \to \mathcal{H}$. Then T is nonexpansive if

$$(\forall x \in D)(\forall y \in D) \quad \|Tx - Ty\|_{\mathcal{H}} \leqslant \|x - y\|_{\mathcal{H}}, \tag{2.5}$$

and firmly nonexpansive if $2T - \mathrm{Id}_{\mathcal{H}}$ is nonexpansive.

The Legendre conjugate of $f\colon \mathcal{H} \to [-\infty, +\infty]$ is the function

$$f^*\colon \mathcal{H} \to [-\infty, +\infty]\colon x^* \mapsto \sup_{x \in \mathcal{H}} \big(\langle x \mid x^*\rangle_{\mathcal{H}} - f(x)\big) \tag{2.6}$$

and the Moreau envelope of $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ of parameter $\gamma \in \left]0, +\infty\right[$ is

$$^{\gamma}\!f\colon \mathcal{H} \to [-\infty, +\infty]\colon x \mapsto \inf_{y \in \mathcal{H}} \Big(f(y) + \frac{1}{2\gamma}\|x - y\|_{\mathcal{H}}^2\Big). \tag{2.7}$$

A function $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ is proper if $\operatorname{dom} f = \big\{x \in \mathcal{H} \mid f(x) < +\infty\big\} \neq \varnothing$. The set of proper lower semicontinuous convex functions from $\mathcal{H}$ to $\left]-\infty, +\infty\right]$ is denoted by $\Gamma_0(\mathcal{H})$. The subdifferential of a function $f \in \Gamma_0(\mathcal{H})$ is

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{x^* \in \mathcal{H} \mid (\forall z \in \mathcal{H})\ \langle z - x \mid x^*\rangle_{\mathcal{H}} + f(x) \leqslant f(z)\big\}, \tag{2.8}$$

and its inverse is

$$(\partial f)^{-1} = \partial f^*. \tag{2.9}$$

Let C be a nonempty convex subset of $\mathcal{H}$. The normal cone of C is denoted by N_C and the strong relative interior of C is denoted by $\operatorname{sri} C$. Additionally, if C is closed, the projection operator onto C is denoted by proj_C. Finally, the closed ball with center $x \in \mathcal{H}$ and radius $\rho \in \left]0, +\infty\right[$ is denoted by $B(x; \rho)$.

The following facts will be used in the paper.

Lemma 2.1. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, let $U\colon \mathcal{G} \to \mathcal{G}$, and let $\rho \in \left]0,+\infty\right[$. Then the following hold:*

(i) $\operatorname{dom}(L \triangleright A) \subset L(\operatorname{dom} A)$.

(ii) *Let $\mathcal{K}$ be a real Hilbert space and let $S \in \mathcal{B}(\mathcal{G},\mathcal{K})$. Then $S \triangleright (L \triangleright A) = (S \circ L) \triangleright A$.*

(iii) $\rho(L \triangleright A) = L \triangleright (\rho A)$.

(iv) $(L \triangleright A)(\rho \mathrm{Id}_{\mathcal{G}}) = L \triangleright (A(\rho \mathrm{Id}_{\mathcal{H}}))$.

(v) $L \triangleright A + U = L \triangleright (A + L^* \circ U \circ L)$.

Proof. (i): By (1.1), $\operatorname{dom}(L \triangleright A) = \operatorname{ran}(L \circ A^{-1} \circ L^*) \subset L(\operatorname{ran} A^{-1}) = L(\operatorname{dom} A)$.

(ii): [7, Proposition 25.42(ii)]

(iii): Since $A^{-1}(\rho^{-1}\mathrm{Id}_{\mathcal{H}}) = (\rho A)^{-1}$, it follows from (1.1) that $\rho(L \triangleright A) = (L \circ A^{-1} \circ L^*(\rho^{-1}\mathrm{Id}_{\mathcal{G}}))^{-1} = (L \circ (A^{-1}(\rho^{-1}\mathrm{Id}_{\mathcal{H}})) \circ L^*)^{-1} = (L \circ (\rho A)^{-1} \circ L^*)^{-1} = L \triangleright (\rho A)$.

(iii): Since $\rho^{-1}A^{-1} = (A(\rho \mathrm{Id}_{\mathcal{H}}))^{-1}$, it follows from (1.1) that $(L \triangleright A)(\rho \mathrm{Id}_{\mathcal{G}}) = (\rho^{-1} L \circ A^{-1} \circ L^*)^{-1} = (L \circ (\rho^{-1}A^{-1}) \circ L^*)^{-1} = (L \circ (A(\rho \mathrm{Id}_{\mathcal{H}}))^{-1} \circ L^*)^{-1} = L \triangleright (A(\rho \mathrm{Id}_{\mathcal{H}}))$.

(v): Let $x \in \mathcal{H}$ and set $M = L \triangleright A + U$. It follows from (1.1) that

$$\begin{aligned}
x^* \in Mx &\Leftrightarrow x^* - Ux \in \big(L \circ A^{-1} \circ L^*\big)^{-1} x \\
&\Leftrightarrow x \in L\Big(A^{-1}\big(L^*x^* - L^*(Ux)\big)\Big) \\
&\Leftrightarrow (\exists\, y \in \mathcal{G}) \quad y \in A^{-1}\big(L^*x^* - L^*(Ux)\big) \quad \text{and} \quad x = Ly \\
&\Leftrightarrow (\exists\, y \in \mathcal{G}) \quad L^*x^* \in Ay + L^*\big(U(Ly)\big) \quad \text{and} \quad x = Ly \\
&\Leftrightarrow (\exists\, y \in \mathcal{G}) \quad y \in \big(A + L^* \circ U \circ L\big)^{-1}(L^*x^*) \quad \text{and} \quad x = Ly \\
&\Leftrightarrow x \in L\Big(\big(A + L^* \circ U \circ L\big)^{-1}(L^*x^*)\Big) \\
&\Leftrightarrow x^* \in \big(L \triangleright (A + L^* \circ U \circ L)\big)x,
\end{aligned} \tag{2.10}$$

which completes the proof. ☐

§3. Resolvent compositions

This section outlines the general properties of the parametrized compositions of Definition 1.1, which will be utilized subsequently.

Proposition 3.1. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, let $\gamma \in \left]0,+\infty\right[$, and let $\rho \in \left]0,+\infty\right[$. Then the following hold:*

(i) $\operatorname{dom}(L \overset{\gamma}{\diamond} B) \subset L^*(\operatorname{dom} B)$.

(ii) $\operatorname{ran}(L \overset{\gamma}{\blacklozenge} B) \subset L^*(\operatorname{ran} B)$.

(iii) $L \overset{\gamma}{\diamond} B = (L \overset{1/\gamma}{\blacklozenge} B^{-1})^{-1}$.

(iv) $\rho(L \overset{\gamma}{\diamond} B) = L \overset{\gamma/\rho}{\diamond} (\rho B)$.

(v) $(L \overset{\gamma}{\diamond} B)(\rho \mathrm{Id}_{\mathcal{H}}) = L \overset{\gamma/\rho}{\diamond} (B(\rho \mathrm{Id}_{\mathcal{G}}))$.

(vi) $\rho(L \overset{\gamma}{\blacklozenge} B) = L \overset{\gamma/\rho}{\blacklozenge} (\rho B)$.

(vii) $(L \overset{\gamma}{\blacklozenge} B)(\rho \mathrm{Id}_{\mathcal{H}}) = L \overset{\gamma/\rho}{\blacklozenge} (B(\rho \mathrm{Id}_{\mathcal{G}}))$.

(viii) *Let $z \in \mathcal{H}$, let $w \in \mathcal{G}$, and set $\tau_w B \colon x \mapsto B(x-w)$. Then the following hold:*

(a) $(L \overset{\gamma}{\diamond} B) - z = L \overset{\gamma}{\diamond} (B - Lz)$.

(b) $\tau_z(L \overset{\gamma}{\blacklozenge} B) = L \overset{\gamma}{\blacklozenge} (\tau_{Lz} B)$.

(ix) *Let $\mathcal{K}$ be a real Hilbert space and $S \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Then the following hold:*

(a) $S \overset{\gamma}{\diamond} (L \overset{\gamma}{\diamond} B) = (L \circ S) \overset{\gamma}{\diamond} B$.

(b) $S \overset{\gamma}{\blacklozenge} (L \overset{\gamma}{\blacklozenge} B) = (L \circ S) \overset{\gamma}{\blacklozenge} B$.

(x) *Set $\beta = \gamma/(1+\rho\gamma)$. Then $L \overset{\gamma}{\diamond} (B + \rho \mathrm{Id}_{\mathcal{G}}) = (L \overset{\beta}{\diamond} B) + \rho \mathrm{Id}_{\mathcal{H}}$.*

(xi) ${}^{\rho}(L \overset{\gamma+\rho}{\blacklozenge} B) = L \overset{\gamma}{\blacklozenge} ({}^{\rho}B)$.

(xii) ${}^{\gamma}(L \overset{\gamma}{\blacklozenge} B) = L^* \circ ({}^{\gamma}B) \circ L$.

(xiii) $\mathrm{zer}(L \overset{\gamma}{\blacklozenge} B) = \mathrm{zer}(L^* \circ ({}^{\gamma}B) \circ L)$.

Proof. (i): By Definition 1.1 and Lemma 2.1(i), $\mathrm{dom}(L \overset{\gamma}{\diamond} B) \subset L^*(\mathrm{dom}(B + \gamma \mathrm{Id}_{\mathcal{G}})) = L^*(\mathrm{dom}\, B)$.

(ii): By Definition 1.1 and (i), $\mathrm{ran}(L \overset{\gamma}{\blacklozenge} B) = \mathrm{dom}(L \overset{1/\gamma}{\diamond} B^{-1}) \subset L^*(\mathrm{dom}\, B^{-1}) = L^*(\mathrm{ran}\, B)$.

(iii): This follows from Definition 1.1.

(iv): It follows from Definition 1.1 and Lemma 2.1(iii) that

$$\begin{aligned}\rho\big(L \overset{\gamma}{\diamond} B\big) &= \rho\big(L^* \triangleright (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}})\big) - \rho\gamma^{-1}\mathrm{Id}_{\mathcal{H}} \\ &= L^* \triangleright (\rho B + \rho\gamma^{-1}\mathrm{Id}_{\mathcal{G}}) - \rho\gamma^{-1}\mathrm{Id}_{\mathcal{H}} \\ &= L \overset{\gamma/\rho}{\diamond} (\rho B). \end{aligned} \tag{3.1}$$

(v): By Definition 1.1 and Lemma 2.1(iv), we obtain

$$\begin{aligned}\big(L \overset{\gamma}{\diamond} B\big)(\rho \mathrm{Id}_{\mathcal{H}}) &= \big(L^* \triangleright (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}})\big)(\rho\mathrm{Id}_{\mathcal{H}}) - \rho\gamma^{-1}\mathrm{Id}_{\mathcal{H}} \\ &= L^* \triangleright \big(B(\rho\mathrm{Id}_{\mathcal{G}}) + \rho\gamma^{-1}\mathrm{Id}_{\mathcal{G}}\big) - \rho\gamma^{-1}\mathrm{Id}_{\mathcal{H}} \\ &= L \overset{\gamma/\rho}{\diamond} \big(B(\rho\mathrm{Id}_{\mathcal{G}})\big). \end{aligned} \tag{3.2}$$

(vi): By Definition 1.1 and (v),

$$\rho\big(L \overset{\gamma}{\blacklozenge} B\big) = \rho\big(L \overset{1/\gamma}{\diamond} B^{-1}\big)^{-1} = \Big(\big(L \overset{1/\gamma}{\diamond} B^{-1}\big)(\mathrm{Id}_{\mathcal{H}}/\rho)\Big)^{-1} = \big(L \overset{\rho/\gamma}{\diamond} (\rho B)^{-1}\big)^{-1} = L \overset{\gamma/\rho}{\blacklozenge} (\rho B). \tag{3.3}$$

(vii): By Definition 1.1 and (iv),

$$\big(L \overset{\gamma}{\blacklozenge} B\big)(\rho\mathrm{Id}_{\mathcal{H}}) = \big(L \overset{1/\gamma}{\diamond} B^{-1}\big)^{-1}(\rho\mathrm{Id}_{\mathcal{H}}) = \Big(\rho^{-1}\big(L \overset{1/\gamma}{\diamond} B^{-1}\big)\Big)^{-1} = \Big(L \overset{\rho/\gamma}{\diamond} \big(B(\rho\mathrm{Id}_{\mathcal{G}})\big)^{-1}\Big)^{-1} = L \overset{\gamma/\rho}{\blacklozenge} \big(B(\rho\mathrm{Id}_{\mathcal{G}})\big). \tag{3.4}$$

(viii)(a): Set $U\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto z$. By Definition 1.1 and Lemma 2.1(v),

$$
\begin{aligned}
\big(L \overset{\gamma}{\diamond} B\big) - z &= L^* \blacktriangleright \big(B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}} - L \circ U \circ L^*\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} \\
&= L^* \blacktriangleright \big(B - Lz + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} \\
&= L \overset{\gamma}{\diamond} (B - Lz). \qquad (3.5)
\end{aligned}
$$

(viii)(b): Since $\tau_w B = (B^{-1} + w)^{-1}$, we combine Definition 1.1 and (viii)(a) to derive

$$
\tau_z\big(L \overset{\gamma}{\blacklozenge} B\big) = \Big(\big(L \overset{1/\gamma}{\diamond} B^{-1}\big) + z\Big)^{-1} = \big(L \overset{1/\gamma}{\diamond} (B^{-1} + Lz)\big)^{-1} = \big(L \overset{1/\gamma}{\diamond} (\tau_{Lz}B)^{-1}\big)^{-1} = L \overset{\gamma}{\blacklozenge} (\tau_{Lz}B). \qquad (3.6)
$$

(ix)(a): By Definition 1.1 and Lemma 2.1(ii),

$$
\begin{aligned}
(L \circ S) \overset{\gamma}{\diamond} B &= (L \circ S)^* \blacktriangleright \big(B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{K}} \\
&= S^* \blacktriangleright \big(L^* \blacktriangleright (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}})\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{K}} \\
&= S^* \blacktriangleright \big((L \overset{\gamma}{\diamond} B) + \gamma^{-1}\mathrm{Id}_{\mathcal{H}}\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{K}} \\
&= S \overset{\gamma}{\diamond} \big(L \overset{\gamma}{\diamond} B\big). \qquad (3.7)
\end{aligned}
$$

(ix)(b): We combine Definition 1.1 and (ix)(a) to obtain

$$
S \overset{\gamma}{\blacklozenge} \big(L \overset{\gamma}{\blacklozenge} B\big) = \Big(S \overset{1/\gamma}{\diamond} \big(L \overset{1/\gamma}{\diamond} B^{-1}\big)\Big)^{-1} = \Big((L \circ S) \overset{1/\gamma}{\diamond} B^{-1}\Big)^{-1} = (L \circ S) \overset{\gamma}{\blacklozenge} B. \qquad (3.8)
$$

(x): Since $\beta^{-1} = \gamma^{-1} + \rho$, we deduce from Definition 1.1 that

$$
\begin{aligned}
L \overset{\gamma}{\diamond} (B + \rho\mathrm{Id}_{\mathcal{G}}) &= L^* \blacktriangleright \big(B + \rho\mathrm{Id}_{\mathcal{G}} + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} \\
&= L^* \blacktriangleright \big(B + \beta^{-1}\mathrm{Id}_{\mathcal{G}}\big) - \beta^{-1}\mathrm{Id}_{\mathcal{H}} + \rho\mathrm{Id}_{\mathcal{H}} \\
&= \big(L \overset{\beta}{\diamond} B\big) + \rho\mathrm{Id}_{\mathcal{H}}. \qquad (3.9)
\end{aligned}
$$

(xi): By (2.2), Definition 1.1, and (x),

$$
\begin{aligned}
{}^{\rho}\big(L \overset{\gamma+\rho}{\blacklozenge} B\big) &= \Big(\big(L \overset{1/(\gamma+\rho)}{\diamond} B^{-1}\big) + \rho\mathrm{Id}_{\mathcal{H}}\Big)^{-1} \\
&= \big(L \overset{1/\gamma}{\diamond} (B^{-1} + \rho\mathrm{Id}_{\mathcal{G}})\big)^{-1} \\
&= L \overset{\gamma}{\blacklozenge} (B^{-1} + \rho\mathrm{Id}_{\mathcal{G}})^{-1} \\
&= L \overset{\gamma}{\blacklozenge} ({}^{\rho}B). \qquad (3.10)
\end{aligned}
$$

(xii): It follows from (2.2) and Definition 1.1 that

$$
{}^{\gamma}\big(L \overset{\gamma}{\blacklozenge} B\big) = \Big(\big(L \overset{1/\gamma}{\diamond} B^{-1}\big) + \gamma\mathrm{Id}_{\mathcal{H}}\Big)^{-1} = \big(L^* \blacktriangleright (B^{-1} + \gamma\mathrm{Id}_{\mathcal{G}})\big)^{-1} = L^* \circ ({}^{\gamma}B) \circ L. \qquad (3.11)
$$

(xiii): Set $A = L \overset{\gamma}{\blacklozenge} B$. It follows from (xii), (2.2), and (2.1) that $0 \in \mathrm{zer}(L^* \circ ({}^{\gamma}B) \circ L) \Leftrightarrow 0 \in \mathrm{zer}({}^{\gamma}A) \Leftrightarrow x \in J_{\gamma A}x \Leftrightarrow 0 \in Ax \Leftrightarrow x \in \mathrm{zer}\, A$. ☐

The following proposition shows that the resolvent of the operators of Definition 1.1 can be computed explicitly.

Proposition 3.2. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, and let $\gamma \in \left]0,+\infty\right[$. Then the following hold:*

(i) $J_{\gamma(L \overset{\gamma}{\diamond} B)} = L^* \circ J_{\gamma B} \circ L$.

(ii) $J_{\gamma(L \overset{\gamma}{\blacklozenge} B)} = \mathrm{Id}_{\mathcal{H}} - L^* \circ (\mathrm{Id}_{\mathcal{G}} - J_{\gamma B}) \circ L$.

Proof. (i): By (2.1), Proposition 3.1(iv), and Definition 1.1,

$$J_{\gamma(L \overset{\gamma}{\diamond} B)} = \Big(\mathrm{Id}_{\mathcal{H}} + \gamma\big(L \overset{\gamma}{\diamond} B\big)\Big)^{-1} = \Big(\mathrm{Id}_{\mathcal{H}} + \big(L \diamond (\gamma B)\big)\Big)^{-1} = \big(L^* \triangleright (\gamma B + \mathrm{Id}_{\mathcal{G}})\big)^{-1} = L^* \circ J_{\gamma B} \circ L. \tag{3.12}$$

(ii): By Proposition 3.1(vi), Definition 1.1, (2.3), and (i), we obtain

$$J_{\gamma(L \overset{\gamma}{\blacklozenge} B)} = J_{L \blacklozenge (\gamma B)} = J_{\left(L \diamond (\gamma B)^{-1}\right)^{-1}} = \mathrm{Id}_{\mathcal{H}} - J_{L \diamond (\gamma B)^{-1}} = \mathrm{Id}_{\mathcal{H}} - L^* \circ (\mathrm{Id}_{\mathcal{G}} - J_{\gamma B}) \circ L, \tag{3.13}$$

as claimed. ☐

The next result interprets resolvent compositions as parallel compositions of perturbed operators.

Proposition 3.3. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, let $\gamma \in \left]0,+\infty\right[$, and set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$. Then the following hold:*

(i) $L \overset{\gamma}{\diamond} B = L^* \triangleright (B + \gamma^{-1}\Psi)$.

(ii) $L \overset{\gamma}{\blacklozenge} B = L^* \circ (B^{-1} + \gamma\Psi)^{-1} \circ L$.

Proof. (i): Combining Definition 1.1 and Lemma 2.1(v), we deduce that

$$\begin{aligned} L \overset{\gamma}{\diamond} B &= L^* \triangleright \big(B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} \\ &= L^* \triangleright \big(B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}} + L \circ (-\gamma^{-1}\mathrm{Id}_{\mathcal{H}}) \circ L^*\big) \\ &= L^* \triangleright \big(B + \gamma^{-1}\Psi\big). \end{aligned} \tag{3.14}$$

(ii): It follows from Definition 1.1 and (i) that

$$L \overset{\gamma}{\blacklozenge} B = \big(L \overset{1/\gamma}{\diamond} B^{-1}\big)^{-1} = \big(L^* \triangleright (B^{-1} + \gamma\Psi)\big)^{-1} = L^* \circ (B^{-1} + \gamma\Psi)^{-1} \circ L, \tag{3.15}$$

as announced. ☐

We proceed to provide particular instances in which the standard, parallel, and resolvent compositions coincide.

Proposition 3.4. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, and let $\gamma \in \left]0,+\infty\right[$. Then the following hold:*

(i) *Suppose that L is an isometry. Then $L \overset{\gamma}{\diamond} B = L \overset{\gamma}{\blacklozenge} B$.*

(ii) *Suppose that L is a coisometry. Then $L \overset{\gamma}{\diamond} B = L^* \triangleright B$ and $L \overset{\gamma}{\blacklozenge} B = L^* \circ B \circ L$.*

(iii) *Suppose that L is invertible with $L^{-1} = L^*$. Then $L \overset{\gamma}{\diamond} B = L^* \triangleright B = L^* \circ B \circ L = L \overset{\gamma}{\blacklozenge} B$.*

Proof. (i): Since $L^* \circ L = \mathrm{Id}_{\mathcal{H}}$, we deduce from Proposition 3.2 and (2.1) that

$$\gamma(L \overset{\gamma}{\blacklozenge} B) = \Big(J_{\gamma(L \overset{\gamma}{\blacklozenge} B)}\Big)^{-1} - \mathrm{Id}_{\mathcal{H}} = \big(L^* \circ J_{\gamma B} \circ L\big)^{-1} - \mathrm{Id}_{\mathcal{H}} = \Big(J_{\gamma(L \overset{\gamma}{\diamond} B)}\Big)^{-1} - \mathrm{Id}_{\mathcal{H}} = \gamma(L \overset{\gamma}{\diamond} B). \tag{3.16}$$

(ii): Since L is a coisometry, $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^* = 0$. Therefore, we derive from Proposition 3.3 that $L \overset{\gamma}{\diamond} B = L^* \triangleright B$ and $L \overset{\gamma}{\blacklozenge} B = L^* \circ B \circ L$.

(iii): This follows from (i) and (ii). ☐

Corollary 3.5. *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, and let $\gamma \in \left]0, +\infty\right[$. Let $\mathcal{K}$ be a real Hilbert space and suppose that $M \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $S \in \mathcal{B}(\mathcal{G})$ are coisometries. Then the following hold:*

(i) $M^* \triangleright (L \overset{\gamma}{\diamond} B) = (L \circ M) \overset{\gamma}{\diamond} B$.

(ii) $M^* \circ (L \overset{\gamma}{\blacklozenge} B) \circ M = (L \circ M) \overset{\gamma}{\blacklozenge} B$.

(iii) $L \overset{\gamma}{\diamond} (S^* \triangleright B) = (S \circ L) \overset{\gamma}{\diamond} B$.

(iv) $L \overset{\gamma}{\blacklozenge} (S^* \circ B \circ S) = (S \circ L) \overset{\gamma}{\blacklozenge} B$

Proof. Combine Proposition 3.4(ii) and Proposition 3.1(ix). ☐

Now, we present several examples of resolvent compositions and cocompositions, starting with the representation of the Yosida approximation as one such composition.

Example 3.6. Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $\gamma \in \left]0, +\infty\right[$. Set $L = \mathrm{Id}_{\mathcal{H}}/2$ and $B = 2A(2\mathrm{Id}_{\mathcal{H}})$. Then $L \overset{\gamma/3}{\blacklozenge} B = {}^{\gamma}\!A$.

Proof. It follows from Proposition 3.3(ii) that $L \overset{4\gamma/3}{\blacklozenge} A = (1/2)(A^{-1} + \gamma \mathrm{Id}_{\mathcal{H}})^{-1}(\mathrm{Id}_{\mathcal{H}}/2) = (1/2)\,{}^{\gamma}\!A(\mathrm{Id}_{\mathcal{H}}/2)$. Therefore, Proposition 3.1(vi)–(vii) implies that ${}^{\gamma}\!A = 2(L \overset{4\gamma/3}{\blacklozenge} A)(2\mathrm{Id}_{\mathcal{H}}) = L \overset{\gamma/3}{\blacklozenge} B$, as claimed. ☐

Example 3.7. Let V be a closed vector subspace of $\mathcal{H}$, $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, and $\gamma \in \left]0, +\infty\right[$. Suppose that L is surjective and that $L^* \circ L = \mathrm{proj}_V$. Then $L \overset{\gamma}{\diamond} B = L^* \triangleright B$, $L \overset{\gamma}{\blacklozenge} B = L^* \circ B \circ L$, and ${}^{\gamma}(L^* \circ B \circ L) = L^* \circ ({}^{\gamma}B) \circ L$.

Proof. In this case, L is a coisometry. Therefore, Proposition 3.4(ii) implies that $L \overset{\gamma}{\diamond} B = L^* \triangleright B$ and $L \overset{\gamma}{\blacklozenge} B = L^* \circ B \circ L$. Further, we use Proposition 3.1(xii) to deduce that ${}^{\gamma}(L^* \circ B \circ L) = {}^{\gamma}(L \overset{\gamma}{\blacklozenge} B) = L^* \circ ({}^{\gamma}B) \circ L$, which completes the proof. ☐

Example 3.8. Let $S \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, let $A\colon \mathcal{G} \to 2^{\mathcal{G}}$, let $\gamma \in \left]0, +\infty\right[$, and let $\mu \in \left]0, +\infty\right[$. Suppose that $S \circ S^* = \mu \mathrm{Id}_{\mathcal{G}}$. Then the following hold:

(i) Set $L = S/\sqrt{\mu}$ and $B = \sqrt{\mu} A(\sqrt{\mu}\mathrm{Id}_{\mathcal{G}})$. Then $L \overset{\gamma}{\blacklozenge} B = S^* \circ A \circ S$.

(ii) $J_{\gamma S^* \circ A \circ S} = \mathrm{Id}_{\mathcal{H}} - \mu^{-1} S^* \circ (\mathrm{Id}_{\mathcal{G}} - J_{\mu\gamma A}) \circ S$.

Proof. (i): In this case, L is a coisometry and Proposition 3.4(ii) yields $L \overset{\gamma}{\blacklozenge} B = L^* \circ B \circ L = S^* \circ A \circ S$.

(ii): By (i), $S^* \circ A \circ S = L \overset{\gamma}{\blacklozenge} B$. Therefore, the result follows from Proposition 3.2(ii) and basic resolvent calculus. ☐

Example 3.9. Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $\|L\| < 1$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, and let $\gamma \in \left]0, +\infty\right[$. Let $\mathcal{X} = \mathcal{H} \oplus \mathcal{G}$, set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$, and set $L_{\Psi}\colon \mathcal{X} \to \mathcal{G}\colon (x, y) \mapsto Lx + \Psi^{1/2} y$. Then

$$L_{\Psi} \overset{\gamma}{\blacklozenge} B\colon \mathcal{X} \to 2^{\mathcal{X}}\colon (x, y) \mapsto \Big(L^*\big(B(Lx + \Psi^{1/2} y)\big)\Big) \times \Big(\Psi^{1/2}\big(B(Lx + \Psi^{1/2} y)\big)\Big). \tag{3.17}$$

Proof. Note that Ψ is self-adjoint and that

$$(\forall y \in \mathcal{G}) \quad \langle \Psi y \mid y \rangle_{\mathcal{G}} = \|y\|_{\mathcal{G}}^2 - \|L^* y\|_{\mathcal{H}}^2 \geqslant (1 - \|L\|^2)\|y\|_{\mathcal{G}}^2. \tag{3.18}$$

Thus, Ψ is strongly monotone and $\Psi^{1/2}$ is well defined. Further, since $L_\Psi^* \colon \mathcal{G} \to \mathcal{X} \colon y \mapsto (L^* y, \Psi^{1/2} y)$, we deduce that

$$(\forall y \in \mathcal{G}) \quad L_\Psi(L_\Psi^* y) = L_\Psi(L^* y, \Psi^{1/2} y) = L(L^* y) + \Psi y = y. \tag{3.19}$$

Therefore, L_Ψ is a coisometry, and it follows from Proposition 3.4(ii) that $L_\Psi \overset{\gamma}{\blacklozenge} B = L_\Psi^* \circ B \circ L_\Psi$, which establishes (3.17). ☐

Example 3.10 (resolvent mixture). Let $0 \neq p \in \mathbb{N}$ and let $\gamma \in \left]0, +\infty\right[$. For every $k \in \{1, \ldots, p\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $L_k \in \mathcal{B}(\mathcal{H}, \mathcal{G}_k)$, let $B_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$, and let $\alpha_k \in \left]0, +\infty\right[$. Let $\mathcal{G} = \bigoplus_{k=1}^p \mathcal{G}_k$, and set

$$L \colon \mathcal{H} \to \mathcal{G} \colon x \mapsto \big(\sqrt{\alpha_1} L_1 x, \ldots, \sqrt{\alpha_p} L_p x\big) \tag{3.20}$$

and

$$B \colon \mathcal{G} \to 2^{\mathcal{G}} \colon (y_k)_{1 \leqslant k \leqslant p} \mapsto \Big(\sqrt{\alpha_1} B_1\big(y_1/\sqrt{\alpha_1}\big)\Big) \times \cdots \times \Big(\sqrt{\alpha_p} B_p\big(y_p/\sqrt{\alpha_p}\big)\Big). \tag{3.21}$$

Then Proposition 3.2 yields

$$J_{\gamma(L \overset{\gamma}{\diamond} B)} = \sum_{k=1}^p \alpha_k L_k^* \circ J_{\gamma B_k} \circ L_k \tag{3.22}$$

and

$$J_{\gamma(L \overset{\gamma}{\blacklozenge} B)} = \mathrm{Id}_{\mathcal{H}} - \sum_{k=1}^p \alpha_k L_k^* \circ (\mathrm{Id}_{\mathcal{G}_k} - J_{\gamma B_k}) \circ L_k. \tag{3.23}$$

The operators

$$\overset{\diamond}{\mathrm{M}}_\gamma(B_k, L_k)_{1 \leqslant k \leqslant p} = L \overset{\gamma}{\diamond} B = \left(\sum_{k=1}^p \alpha_k L_k^* \circ \big(B_k + \gamma^{-1}\mathrm{Id}_{\mathcal{G}_k}\big)^{-1} \circ L_k\right)^{-1} - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} \tag{3.24}$$

and

$$\overset{\blacklozenge}{\mathrm{M}}_\gamma(B_k, L_k)_{1 \leqslant k \leqslant p} = L \overset{\gamma}{\blacklozenge} B = \left(\left(\sum_{k=1}^p \alpha_k L_k^* \circ \big(B_k^{-1} + \gamma \mathrm{Id}_{\mathcal{G}_k}\big)^{-1} \circ L_k\right)^{-1} - \gamma \mathrm{Id}_{\mathcal{H}}\right)^{-1} \tag{3.25}$$

are called the *resolvent mixture* and *comixture*, respectively, and were initially introduced in [11, Example 3.4] (see also [8] for further developments).

Example 3.11 (resolvent average). In the context of Example 3.10, suppose that $\sum_{k=1}^p \alpha_k = 1$ and that, for every $k \in \{1, \ldots, p\}$, $\mathcal{G}_k = \mathcal{H}$ and $L_k = \mathrm{Id}_{\mathcal{H}}$. Since $L^* \colon \mathcal{G} \to \mathcal{H} \colon (y_k)_{1 \leqslant k \leqslant p} \mapsto \sum_{k=1}^p \sqrt{\alpha_k} y_k$, the linear operator L is an isometry. Thus, Proposition 3.4(i) yields $L \overset{\gamma}{\diamond} B = L \overset{\gamma}{\blacklozenge} B$. This operator is called the *resolvent average* of $(B_k)_{1 \leqslant k \leqslant p}$ and $(\alpha_k)_{1 \leqslant k \leqslant p}$, denoted by $\mathrm{rav}_\gamma(B_k, \alpha_k)_{1 \leqslant k \leqslant p}$, which has been studied in [6, 7, 8, 11, 23], namely,

$$L \overset{\gamma}{\diamond} B = \left(\sum_{k=1}^p \alpha_k \big(B_k + \gamma^{-1}\mathrm{Id}_{\mathcal{H}}\big)^{-1}\right)^{-1} - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} = \mathrm{rav}_\gamma(B_k, \alpha_k)_{1 \leqslant k \leqslant p}. \tag{3.26}$$

In addition, (3.22) yields $J_{\gamma\, \mathrm{rav}_\gamma(B_k, \alpha_k)_{1 \leqslant k \leqslant p}} = \sum_{k=1}^p \alpha_k J_{\gamma B_k}$.

§4. Monotonicity of resolvent compositions

We leverage the results of Section 3 to establish conditions that ensure the monotonicity of resolvent compositions.

Proposition 4.1. *Suppose that* $0 \neq L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, *let* $B \colon \mathcal{G} \to 2^{\mathcal{G}}$, *let* $\gamma \in \left]0, +\infty\right[$, *let* $\alpha \in \left[-1/\gamma, +\infty\right[$, *and set* $\beta = (\alpha + \gamma^{-1})\|L\|^{-2} - \gamma^{-1}$. *Suppose that* $B - \alpha \mathrm{Id}_{\mathcal{G}}$ *is monotone. Then* $L \overset{\gamma}{\diamond} B - \beta \mathrm{Id}_{\mathcal{H}}$ *is monotone.*

Proof. Set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$ and $\mathcal{M} = (B - \alpha \mathrm{Id}_{\mathcal{G}}) + (\alpha + \gamma^{-1})\|L\|^{-2}(\|L\|^2 \mathrm{Id}_{\mathcal{G}} - L \circ L^*)$. It follows from Proposition 3.3(i) and Lemma 2.1(v) that

$$
\begin{aligned}
\big(L \overset{\gamma}{\diamond} B\big) - \beta \mathrm{Id}_{\mathcal{H}} &= L^* \triangleright \big(B + \gamma^{-1}\Psi\big) - \beta \mathrm{Id}_{\mathcal{H}} \\
&= L^* \triangleright \big(B + \gamma^{-1}\Psi + L \circ (-\beta \mathrm{Id}_{\mathcal{H}}) \circ L^*\big) \\
&= L^* \triangleright \big((B - \alpha \mathrm{Id}_{\mathcal{G}}) + (\alpha + \gamma^{-1})\|L\|^{-2}(\|L\|^2 \mathrm{Id}_{\mathcal{G}} - L \circ L^*)\big) \\
&= L^* \triangleright \mathcal{M}. \qquad (4.1)
\end{aligned}
$$

Since $\alpha + \gamma^{-1} \geqslant 0$ and the operators $B - \alpha \mathrm{Id}_{\mathcal{G}}$ and $\|L\|^2 \mathrm{Id}_{\mathcal{G}} - L \circ L^*$ are monotone, [7, Propositions 20.10] implies that $\mathcal{M}$ is monotone. Therefore, the assertion follows from (4.1) and [7, Proposition 25.41(ii)]. □

Corollary 4.2. *Suppose that* $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $B \colon \mathcal{G} \to 2^{\mathcal{G}}$, *let* $\gamma \in \left]0, +\infty\right[$, *let* $\alpha \in \left]0, +\infty\right[$, *and set* $\beta = (\alpha + \gamma^{-1})\|L\|^{-2} - \gamma^{-1}$. *Suppose that* B *is* α*-strongly monotone. Then* $L \overset{\gamma}{\diamond} B$ *is* β*-strongly monotone.*

Proof. Since $\beta = \alpha\|L\|^{-2} + \gamma^{-1}(\|L\|^{-2} - 1) > 0$, the conclusion follows from Proposition 4.1. □

Corollary 4.3. *Suppose that* $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ *satisfies* $0 < \|L\| < 1$, *let* $B \colon \mathcal{G} \to 2^{\mathcal{G}}$ *be monotone, let* $\gamma \in \left]0, +\infty\right[$, *and set* $\beta = \gamma^{-1}(\|L\|^{-2} - 1)$. *Then* $L \overset{\gamma}{\diamond} B$ *is* β*-strongly monotone.*

Proof. This follows from Proposition 4.1 when $\alpha = 0$. □

Proposition 4.4. *Let* $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, $B \colon \mathcal{G} \to 2^{\mathcal{G}}$, *and* $\gamma \in \left]0, +\infty\right[$. *Suppose that* $B + \gamma^{-1}(\mathrm{Id}_{\mathcal{G}} - L \circ L^*)$ *is monotone. Then the following hold:*

(i) $L \overset{\gamma}{\diamond} B$ *is monotone.*

(ii) *Suppose that* $\operatorname{ran} L \subset \operatorname{ran}(\mathrm{Id}_{\mathcal{G}} + \gamma B)$. *Then* $L \overset{\gamma}{\diamond} B$ *is maximally monotone.*

Proof. Set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$ and recall from Proposition 3.3(i) that $L \overset{\gamma}{\diamond} B = L^* \triangleright (B + \gamma^{-1}\Psi)$.

(i): Since $B + \gamma^{-1}\Psi$ is monotone, we deduce from [7, Proposition 25.41(ii)] that $L \overset{\gamma}{\diamond} B$ is monotone.

(ii): Since monotonicity is preserved under multiplication by positive scalars, $\gamma(L \overset{\gamma}{\diamond} B)$ is monotone by (i). Further, by assumption, $\operatorname{ran} L \subset \operatorname{ran}(\mathrm{Id}_{\mathcal{G}} + \gamma B) = \operatorname{dom}(\mathrm{Id}_{\mathcal{G}} + \gamma B)^{-1} = \operatorname{dom} J_{\gamma B}$. Therefore, $\operatorname{dom}(L^* \circ J_{\gamma B} \circ L) = \operatorname{dom}(J_{\gamma B} \circ L) = \mathcal{H}$, and it follows from Proposition 3.2(i) that $\operatorname{ran}(\mathrm{Id}_{\mathcal{H}} + \gamma(L \overset{\gamma}{\diamond} B)) = \operatorname{dom} J_{\gamma(L \overset{\gamma}{\diamond} B)} = \mathcal{H}$. Altogether, we deduce from [7, Theorem 21.1 (Minty)] that $\gamma(L \overset{\gamma}{\diamond} B)$ is maximally monotone. Hence, $L \overset{\gamma}{\diamond} B$ is maximally monotone. □

The following result recovers [11, Proposition 4.4(i)–(ii) and Theorem 4.5(i)–(ii)], which were proven when $\gamma = 1$ using distinct approaches.

Corollary 4.5. *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $\|L\| \leqslant 1$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be monotone, and let $\gamma \in \left]0, +\infty\right[$. Then the following hold:*

(i) *$L \overset{\gamma}{\diamond} B$ and $L \overset{\gamma}{\blacklozenge} B$ are monotone.*

(ii) *Suppose that B is maximally monotone. Then $L \overset{\gamma}{\diamond} B$ and $L \overset{\gamma}{\blacklozenge} B$ are maximally monotone.*

Proof. Set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$. Since $\|L\| \leqslant 1$, Ψ is monotone. Thus, [7, Proposition 20.10] implies that $B + \gamma^{-1}\Psi$ and B^{-1} are monotone.

(i): By Proposition 4.4(i), $L \overset{\gamma}{\diamond} B$ and $L \overset{1/\gamma}{\diamond} B^{-1}$ are monotone. Therefore, we combine Definition 1.1 and [7, Proposition 20.10] to deduce that $L \overset{\gamma}{\blacklozenge} B = (L \overset{1/\gamma}{\diamond} B^{-1})^{-1}$ is monotone.

(ii): By [7, Proposition 20.22], γB and B^{-1} are maximally monotone. Therefore, [7, Theorem 21.1] yields $\mathrm{ran}(\mathrm{Id}_{\mathcal{G}} + \gamma B) = \mathcal{G}$, and, by Proposition 4.4(ii), $L \overset{\gamma}{\diamond} B$ is maximally monotone. Similarly, $L \overset{1/\gamma}{\diamond} B^{-1}$ is maximally monotone, and it follows from [7, Proposition 20.22] that $L \overset{\gamma}{\blacklozenge} B = (L \overset{1/\gamma}{\diamond} B^{-1})^{-1}$ is maximally monotone as well. ☐

Next, we state several instantiations of resolvent compositions which are monotone.

Example 4.6. Let $0 \neq p \in \mathbb{N}$ and let $\gamma \in \left]0, +\infty\right[$. For every $k \in \{1, \ldots, p\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $L_k \in \mathcal{B}(\mathcal{H}, \mathcal{G}_k)$, let $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ be maximally monotone, and let $\alpha_k \in \left]0, +\infty\right[$. Suppose that $\sum_{k=1}^{p} \alpha_k \|L_k\|^2 \leqslant 1$. Then $\overset{\diamond}{\mathrm{M}}_{\gamma}(B_k, L_k)_{1 \leqslant k \leqslant p}$ and $\overset{\blacklozenge}{\mathrm{M}}_{\gamma}(B_k, L_k)_{1 \leqslant k \leqslant p}$ are maximally monotone.

Proof. Define L as in (3.20) and B as in (3.21). Thus, the assumption $\sum_{k=1}^{p} \alpha_k \|L_k\|^2 \leqslant 1$ implies that $\|L\| \leqslant 1$, and, by [7, Proposition 23.18], B is maximally monotone. Since $\overset{\diamond}{\mathrm{M}}_{\gamma}(B_k, L_k)_{1 \leqslant k \leqslant p} = L \overset{\gamma}{\diamond} B$ and $\overset{\blacklozenge}{\mathrm{M}}_{\gamma}(B_k, L_k)_{1 \leqslant k \leqslant p} = L \overset{\gamma}{\blacklozenge} B$, the conclusion follows from Corollary 4.5(ii). ☐

Example 4.7. Let $A_1\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $A_2\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, let $\mathcal{G} = \mathcal{H} \oplus \mathcal{H}$, set $L\colon \mathcal{H} \to \mathcal{G}\colon x \mapsto (x, -x)$, and set

$$B\colon \mathcal{G} \to 2^{\mathcal{G}}\colon (x, y) \mapsto \big(A_1 x\big) \times \big(A_2^{-1} y - 2x\big). \tag{4.2}$$

Then $L \diamond B$ is maximally monotone and

$$J_{L \diamond B} = \frac{1}{2}\mathrm{Id}_{\mathcal{H}} + \frac{1}{2}\big(2J_{A_2} - \mathrm{Id}_{\mathcal{H}}\big) \circ \big(2J_{A_1} - \mathrm{Id}_{\mathcal{H}}\big). \tag{4.3}$$

In other words, the resolvent of $L \diamond B$ is the *Douglas–Rachford splitting operator* of A_2 and A_1 (see [7, 18]).

Proof. Set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$, $\mathcal{M}\colon \mathcal{G} \to 2^{\mathcal{G}}\colon (x, y) \mapsto (A_1 x) \times (A_2^{-1} y)$, and $S\colon \mathcal{G} \to \mathcal{G}\colon (x, y) \mapsto (y, -x)$. Note that $\Psi\colon (x, y) \mapsto (y, x)$, that S is monotone, and that, since A_1 and A_2 are monotone, $\mathcal{M}$ is monotone as well. Thus, [7, Proposition 20.10] implies that $B + \Psi = \mathcal{M} + S$ is monotone. Further, it is straightforward verify that

$$(\forall (x, y) \in \mathcal{G}) \quad J_B(x, y) = \big(J_{A_1} x, J_{A_2^{-1}}(y + 2J_{A_1} x)\big). \tag{4.4}$$

Therefore, it follows from Proposition 4.4(ii) that $L \diamond B$ is maximally monotone. In addition, since $L^*: \mathcal{G} \to \mathcal{H}: (x,y) \mapsto x - y$, Proposition 3.2(i), (4.4), and (2.3) yield

$$
\begin{aligned}
(\forall x \in \mathcal{H}) \quad J_{L \diamond B} x &= L^*\big(J_B(x,-x)\big) \\
&= L^*\big(J_{A_1}x, J_{A_2^{-1}}(2J_{A_1}x - x)\big) \\
&= J_{A_1}x - J_{A_2^{-1}}(2J_{A_1}x - x) \\
&= J_{A_1}x - \big(2J_{A_1}x - x - J_{A_2}(2J_{A_1}x - x)\big) \\
&= \frac{1}{2}x + \frac{1}{2}\big(2J_{A_2} - \mathrm{Id}_{\mathcal{H}}\big)(2J_{A_1}x - x), \qquad (4.5)
\end{aligned}
$$

which establishes (4.3). ☐

Remark 4.8. Consider the setting of Example 4.7. Then the operator B is not necessarily monotone (take $A_1 = 0$ and $A_2 = N_{\{0\}}$) and the norm of the linear operator is greater than 1 ($\|L\| = \sqrt{2}$). As a result, the resolvent composition $L \diamond B$ can be maximally monotone, even in cases when B is not monotone and $\|L\| > 1$.

The following example recovers the operator used in [20] for finding a zero in the sum of $p \geqslant 2$ maximally monotone operators. For the sake of simplicity, we represent operators using matrices.

Example 4.9. Let $\gamma \in \left]0,+\infty\right[$, let $\mathcal{K}$ be a real Hilbert space, let $p \in \mathbb{N} \setminus \{0,1\}$, and let $(A_k)_{1 \leqslant k \leqslant p}$ be a family of maximally monotone operators in $\mathcal{K}$. Let $\mathcal{H} = \bigoplus_{k=1}^{p-1} \mathcal{K}$, let $\mathcal{G} = \bigoplus_{k=1}^{p} \mathcal{K}$, set

$$
L = \begin{bmatrix}
\mathrm{Id}_{\mathcal{K}} & & & \\
-\mathrm{Id}_{\mathcal{K}} & \mathrm{Id}_{\mathcal{K}} & & \\
 & \ddots & \ddots & \\
 & & -\mathrm{Id}_{\mathcal{K}} & \mathrm{Id}_{\mathcal{K}} \\
 & & & -\mathrm{Id}_{\mathcal{K}}
\end{bmatrix} \in \mathcal{B}(\mathcal{H},\mathcal{G}), \qquad (4.6)
$$

and set

$$
B = \gamma \begin{bmatrix}
A_1 & & & & \\
-\mathrm{Id}_{\mathcal{K}} & A_2 & & & \\
 & \ddots & \ddots & & \\
 & & -\mathrm{Id}_{\mathcal{K}} & A_{p-1} & \\
-\mathrm{Id}_{\mathcal{K}} & & & -\mathrm{Id}_{\mathcal{K}} & A_p
\end{bmatrix} : \mathcal{G} \to 2^{\mathcal{G}}. \qquad (4.7)
$$

Then, for every $z = (z_k)_{1 \leqslant k \leqslant p-1} \in \mathcal{H}$,

$$
J_{\gamma\left((\gamma L) \overset{\gamma}{\diamond} B^{-1}\right)} z = z + \gamma^2 \begin{bmatrix}
x_2 - x_1 \\
x_3 - x_2 \\
\vdots \\
x_p - x_{p-1}
\end{bmatrix}, \quad \text{where} \quad
\begin{cases}
x_1 = J_{A_1} z_1, \\
(\forall k \in \{2,\ldots,p-1\}) \;\; x_k = J_{A_k}(z_k + x_{k-1} - z_{k-1}), \\
x_p = J_{A_p}(x_1 + x_{p-1} - z_{p-1}).
\end{cases} \qquad (4.8)
$$

Proof. Let $\boldsymbol{z} = (z_k)_{1\leqslant k\leqslant p-1} \in \boldsymbol{\mathcal{H}}$. It is straightforward verify that $J_{\gamma^{-1}\boldsymbol{B}}(\boldsymbol{L}\boldsymbol{z}) = (x_k)_{1\leqslant k\leqslant p} = \boldsymbol{x}$. On the other hand, recall from [7, Proposition 23.20] that $\mathrm{Id}_{\boldsymbol{\mathcal{G}}} - J_{\gamma\boldsymbol{B}^{-1}} = \gamma J_{\gamma^{-1}\boldsymbol{B}}(\gamma^{-1}\mathrm{Id}_{\boldsymbol{\mathcal{G}}})$. Further, note that

$$\boldsymbol{L}^*\boldsymbol{x} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ \vdots \\ x_{p-1} - x_p \end{bmatrix} \tag{4.9}$$

Altogether, we deduce from Proposition 3.2(ii) and (4.9) that

$$J_{\gamma\left((\gamma\boldsymbol{L})\overset{\gamma}{\blacklozenge}\boldsymbol{B}^{-1}\right)}\boldsymbol{z} = \boldsymbol{z} - (\gamma\boldsymbol{L})^*\big(\gamma J_{\gamma^{-1}\boldsymbol{B}}(\boldsymbol{L}\boldsymbol{z})\big) = \boldsymbol{z} - \gamma^2\boldsymbol{L}^*\boldsymbol{x} = \boldsymbol{z} + \gamma^2 \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_p - x_{p-1}, \end{bmatrix}, \tag{4.10}$$

which establishes (4.8). ☐

Remark 4.10. As shown in [20, Lemma 3], the operator given in (4.8) is γ^2-averaged when $\gamma \in \left]0,1\right[$.

Example 4.11. Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ be such that $\operatorname{ran} L$ is closed, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, and let $\gamma \in \left]0,+\infty\right[$. Let $\mathcal{X}$ be the real Hilbert space obtained by endowing $\mathcal{H}$ with the scalar product

$$\langle\cdot\mid\cdot\rangle_{\mathcal{X}}\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}\colon (x,y) \mapsto \langle Lx \mid Ly\rangle_{\mathcal{G}} + \langle x \mid \mathrm{proj}_{\ker L}\, y\rangle_{\mathcal{H}}, \tag{4.11}$$

and set $L_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{G}\colon x \mapsto Lx$. Then the following hold:

(i) $L_{\mathcal{X}} \overset{\gamma}{\diamond} B$ and $L_{\mathcal{X}} \overset{\gamma}{\blacklozenge} B$ are maximally monotone.

(ii) $J_{\gamma(L_{\mathcal{X}}\overset{\gamma}{\diamond}B)} = L^\dagger \circ J_{\gamma B} \circ L$.

(iii) $J_{\gamma(L_{\mathcal{X}}\overset{\gamma}{\blacklozenge}B)} = \mathrm{Id}_{\mathcal{X}} - L^\dagger \circ (\mathrm{Id}_{\mathcal{G}} - J_{\gamma B}) \circ L$.

(iv) Suppose that $\ker L = \{0\}$. Then $L_{\mathcal{X}} \overset{\gamma}{\diamond} B = L_{\mathcal{X}} \overset{\gamma}{\blacklozenge} B$.

(v) Suppose that $\operatorname{ran} L = \mathcal{G}$. Then $L_{\mathcal{X}} \overset{\gamma}{\diamond} B = L^\dagger \rhd B$ and $L_{\mathcal{X}} \overset{\gamma}{\blacklozenge} B = L^\dagger \circ B \circ L$.

Proof. Let $x \in \mathcal{H}$ and $y \in \mathcal{G}$. It follows from (4.11) that $\|L_{\mathcal{X}}\| \leqslant 1$ since

$$(\forall z \in \mathcal{H})\quad \|L_{\mathcal{X}}z\|_{\mathcal{G}}^2 = \|Lz\|_{\mathcal{G}}^2 \leqslant \|Lz\|_{\mathcal{G}}^2 + \|\mathrm{proj}_{\ker L}\, z\|_{\mathcal{H}}^2 = \|z\|_{\mathcal{X}}^2. \tag{4.12}$$

Further, the identities $L^*y = L^*(L(L^\dagger y))$ and $L^\dagger y \in (\ker L)^\perp$ [7, Proposition 3.30(i)] imply that

$$\begin{aligned}\langle L_{\mathcal{X}}x \mid y\rangle_{\mathcal{G}} &= \langle x \mid L^*y\rangle_{\mathcal{H}} \\ &= \langle x \mid L^*\big(L(L^\dagger y)\big)\rangle_{\mathcal{H}} \\ &= \langle Lx \mid L(L^\dagger y)\rangle_{\mathcal{G}} + \langle x \mid \mathrm{proj}_{\ker L}(L^\dagger y)\rangle_{\mathcal{H}} \\ &= \langle x \mid L^\dagger y\rangle_{\mathcal{X}}. \end{aligned} \tag{4.13}$$

In turn, $L_{\mathcal{X}}^*\colon \mathcal{G} \to \mathcal{X}\colon y \mapsto L^\dagger y$.

(i): A consequence of Corollary 4.5(ii).

(ii)–(iii): A consequence of Proposition 3.2.

(iv): By [7, Proposition Corollary 3.32(iv)], $L^\dagger \circ L = \mathrm{Id}_{\mathcal{H}}$. Therefore, $L_{\mathcal{X}}$ is an isometry, and the assertion follows from Proposition 3.4(i).

(v): By [7, Proposition 3.30(ii)], $L \circ L^\dagger = \mathrm{Id}_{\mathcal{G}}$. Therefore, $L_{\mathcal{X}}$ is a coisometry, and the assertion follows from Proposition 3.4(ii). ☐

Remark 4.12. When $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $L^\dagger \circ L = \mathrm{Id}_{\mathcal{H}}$, the operator $T = L^\dagger \circ J_{\gamma\partial\|\cdot\|_{\mathcal{H}}} \circ L$ has been used to enhance the performance of wavelet-domain denoising [9]. Consequently, Example 4.11(ii) shows that this method implicitly involves resolvent compositions.

Example 4.13. Let $U \in \mathcal{B}(\mathcal{H})$ be a self-adjoint and strongly monotone operator. In the context of Example 4.11, assume that $\mathcal{G} = \mathcal{H}$ and that $L = U^{-1/2}$. Then

$$L_{\mathcal{X}} \overset{\gamma}{\diamond} B = U^{1/2} \circ B \circ U^{-1/2}. \tag{4.14}$$

Proof. In this case, L is invertible and $L^\dagger = L^{-1} = U^{1/2}$. Therefore, Example 4.11(iv)–(v) implies that $L_{\mathcal{X}} \overset{\gamma}{\diamond} B = L_{\mathcal{X}} \overset{\gamma}{\blacklozenge} B = U^{1/2} \circ B \circ U^{-1/2}$, as claimed. ☐

Example 4.14. Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $g\colon \mathcal{G} \to \left]-\infty, +\infty\right]$ be a proper function that admits a continuous affine minorant, let $\gamma \in \left]0, +\infty\right[$, let

$$L \overset{\gamma}{\diamond} g = \left(\tfrac{1}{\gamma}(g^*) \circ L\right)^* - \frac{1}{2\gamma}\|\cdot\|_{\mathcal{H}}^2 \tag{4.15}$$

be the *proximal composition* of g and L, and let $L \overset{\gamma}{\blacklozenge} g = (L \overset{1/\gamma}{\diamond} g^*)^*$ be the *proximal cocomposition* of g and L (see [8, 11, 14]). Then the following hold:

(i) $L \overset{\gamma}{\diamond} \partial g^{**} = \partial(L \overset{\gamma}{\diamond} g)$.

(ii) $L \overset{\gamma}{\blacklozenge} \partial g^{**} = \partial(L \overset{\gamma}{\blacklozenge} g)$.

Proof. Set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$ and recall from [14, Lemma 2.1(v)] that $g^* \in \Gamma_0(\mathcal{G})$.

(i): By [14, Proposition 3.11(i)] and Proposition 3.3(i), $\partial(L \overset{\gamma}{\diamond} g) = L^* \triangleright (\partial g^{**} + \gamma^{-1}\Psi) = L \overset{\gamma}{\diamond} \partial g^{**}$.

(ii): Note that (2.9) and the identity $g^{***} = g^*$ yield $(\partial g^{**})^{-1} = \partial g^{***} = \partial g^*$. Therefore, it follows from [14, Proposition 3.11(ii)] and Proposition 3.3(ii) that $\partial(L \overset{\gamma}{\blacklozenge} g) = L^* \circ (\partial g^* + \gamma\Psi)^{-1} \circ L = L \overset{\gamma}{\blacklozenge} \partial g^{**}$, which provides the desired identity. ☐

We conclude this section by examining the resolvent composition of uniformly monotone operators as well as the Fitzpatrick function of resolvent compositions.

Proposition 4.15. *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, and let $\gamma \in \left]0, +\infty\right[$. Suppose that B is uniformly monotone with modulus $\phi\colon \left[0, +\infty\right[\to \left[0, +\infty\right]$, i.e., ϕ is increasing, vanishes only at 0, and*

$$\big(\forall (y_1, y_1^*) \in \operatorname{gra} B\big)\big(\forall (y_2, y_2^*) \in \operatorname{gra} B\big) \quad \langle y_1 - y_2 \mid y_1^* - y_2^* \rangle_{\mathcal{G}} \geqslant \phi(\|y_1 - y_2\|_{\mathcal{G}}), \tag{4.16}$$

and set $\phi_L = L \overset{\gamma/2}{\diamond} (\phi \circ \|\cdot\|_{\mathcal{G}})$. Then

$$\big(\forall (x_1, x_1^*) \in \operatorname{gra}(L \overset{\gamma}{\diamond} B)\big)\big(\forall (x_2, x_2^*) \in \operatorname{gra}(L \overset{\gamma}{\diamond} B)\big) \quad \langle x_1 - x_2 \mid x_1^* - x_2^* \rangle_{\mathcal{G}} \geqslant \phi_L(x_1 - x_2). \tag{4.17}$$

Proof. Note that $\phi \circ \|\cdot\|_{\mathcal{G}} \geqslant 0$ and that $\phi(\|0\|_{\mathcal{G}}) = 0$. Thus, $\phi \circ \|\cdot\|_{\mathcal{G}}$ is a proper function minorized by the affine function 0. Further, by [7, Proposition 13.16], $\phi \circ \|\cdot\|_{\mathcal{G}} \geqslant (\phi \circ \|\cdot\|_{\mathcal{G}})^{**}$. On the other hand, recall from Corollary 4.5(ii) that $L \overset{\gamma}{\diamond} B$ is maximally monotone. Let $(x_1, x_1^*) \in \operatorname{gra}(L \overset{\gamma}{\diamond} B)$ and $(x_2, x_2^*) \in \operatorname{gra}(L \overset{\gamma}{\diamond} B)$. It follows from Proposition 3.2(i) that

$$
\begin{aligned}
(\forall k \in \{1,2\}) \quad x_k^* \in \big(L \overset{\gamma}{\diamond} B\big)x_k &\Leftrightarrow J_{\gamma(L \overset{\gamma}{\diamond} B)}(x_k + \gamma x_k^*) = x_k \\
&\Leftrightarrow \begin{cases} (\exists\, p_k \in \mathcal{G})\ L^* p_k = x_k \\ J_{\gamma B}\big(L(x_k + \gamma x_k^*)\big) = p_k \end{cases} \\
&\Leftrightarrow \begin{cases} (\exists\, p_k \in \mathcal{G})\ L^* p_k = x_k \\ \big(p_k, L(\gamma^{-1} x_k + x_k^*) - \gamma^{-1} p_k\big) \in \operatorname{gra} B. \end{cases}
\end{aligned} \tag{4.18}
$$

Since B is uniformly monotone with modulus ϕ, we deduce that

$$
\begin{aligned}
\gamma^{-1}\|x_1 - x_2\|_{\mathcal{H}}^2 + \langle x_1 - x_2 \mid x_1^* - x_2^*\rangle_{\mathcal{H}} - \gamma^{-1}\|p_1 - p_2\|_{\mathcal{G}}^2 &= \langle p_1 - p_2 \mid \gamma^{-1} L(x_1 - x_2) + L(x_1^* - x_2^*)\rangle_{\mathcal{G}} \\
&\quad - \gamma^{-1}\langle p_1 - p_2 \mid p_1 - p_2\rangle_{\mathcal{G}} \\
&\geqslant \phi(\|p_1 - p_2\|_{\mathcal{G}}).
\end{aligned} \tag{4.19}
$$

Therefore, since $L^*(p_1 - p_2) = x_1 - x_2$, we deduce from (4.19) and [14, Proposition 3.2(i)] that

$$
\begin{aligned}
\langle x_1 - x_2 \mid x_1^* - x_2^*\rangle_{\mathcal{H}} &\geqslant \phi(\|p_1 - p_2\|_{\mathcal{G}}) + \gamma^{-1}\big(\|p_1 - p_2\|_{\mathcal{G}}^2 - \|x_1 - x_2\|_{\mathcal{H}}^2\big) \\
&\geqslant \inf_{\substack{v \in \mathcal{G} \\ L^* v = x_1 - x_2}} \Big(\phi(\|v\|_{\mathcal{G}}) + \gamma^{-1}\big(\|v\|_{\mathcal{G}} - \|L^* v\|_{\mathcal{H}}^2\big)\Big) \\
&\geqslant \inf_{\substack{v \in \mathcal{G} \\ L^* v = x_1 - x_2}} \Big(\big(\phi \circ \|\cdot\|_{\mathcal{G}}\big)^{**}(v) + \gamma^{-1}\big(\|v\|_{\mathcal{G}} - \|L^* v\|_{\mathcal{H}}^2\big)\Big) \\
&= \big(L \overset{\gamma/2}{\diamond} (\phi \circ \|\cdot\|_{\mathcal{G}})\big)(x_1 - x_2),
\end{aligned} \tag{4.20}
$$

which completes the proof. ☐

Proposition 4.16 (Fitzpatrick function). *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $\|L\| \leqslant 1$, let $B \colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, let*

$$
F_B \colon \mathcal{G} \times \mathcal{G} \to [-\infty, +\infty] \colon (x, x^*) \mapsto \sup_{(v, v^*) \in \operatorname{gra} B} \big(\langle v \mid x^*\rangle_{\mathcal{G}} + \langle x \mid v^*\rangle_{\mathcal{G}} - \langle v \mid v^*\rangle_{\mathcal{G}}\big) \tag{4.21}
$$

be its Fitzpatrick function, and let $\gamma \in \,]0, +\infty[$. Then the following hold:

(i) *Let $x \in \ker(\mathrm{Id}_{\mathcal{H}} - L^* \circ L)$ and $x^* \in \mathcal{H}$. Then $F_{\gamma(L \overset{\gamma}{\diamond} B)}(x, x^*) \leqslant F_{\gamma B}(Lx, Lx^*)$.*

(ii) *Let $x \in \mathcal{H}$ and $x^* \in \ker(\mathrm{Id}_{\mathcal{H}} - L^* \circ L)$. Then $F_{\gamma(L \overset{\gamma}{\blacklozenge} B)}(x, x^*) \leqslant F_{\gamma B}(Lx, Lx^*)$.*

Proof. We recall from Corollary 4.5(ii) that $L \overset{\gamma}{\diamond} B$ and $L \overset{\gamma}{\blacklozenge} B$ are maximally monotone.

(i): By Minty's parametrization [7, Remark 23.23(ii)],

$$
F_{\gamma(L \overset{\gamma}{\diamond} B)}(x, x^*) = \sup_{y \in \mathcal{H}} \Big(\Big\langle J_{\gamma(L \overset{\gamma}{\diamond} B)} y \,\Big|\, x^*\Big\rangle_{\mathcal{H}} + \Big\langle x \,\Big|\, y - J_{\gamma(L \overset{\gamma}{\diamond} B)} y\Big\rangle_{\mathcal{H}} - \Big\langle J_{\gamma(L \overset{\gamma}{\diamond} B)} y \,\Big|\, y - J_{\gamma(L \overset{\gamma}{\diamond} B)} y\Big\rangle_{\mathcal{H}}\Big). \tag{4.22}
$$

Thus, by virtue of Proposition 3.2(i) and $\|L\| \leqslant 1$, we deduce that, for every $y \in \mathcal{H}$,

$$\begin{aligned}
&\left\langle x \,\middle|\, y - J_{\gamma(L \overset{\gamma}{\diamond} B)} y \right\rangle_{\mathcal{H}} + \left\langle J_{\gamma(L \overset{\gamma}{\diamond} B)} y \,\middle|\, x^* \right\rangle_{\mathcal{H}} - \left\langle J_{\gamma(L \overset{\gamma}{\diamond} B)} y \,\middle|\, y - J_{\gamma(L \overset{\gamma}{\diamond} B)} y \right\rangle_{\mathcal{H}} \\
&= \left\langle x \,\middle|\, y - L^*(J_{\gamma B}(Ly)) \right\rangle_{\mathcal{H}} + \left\langle L^*(J_{\gamma B}(Ly)) \,\middle|\, x^* \right\rangle_{\mathcal{H}} - \left\langle L^*(J_{\gamma B}(Ly)) \,\middle|\, y - L^*(J_{\gamma B}(Ly)) \right\rangle_{\mathcal{H}} \\
&\leqslant \langle x - L^*(Lx) \mid y \rangle_{\mathcal{H}} + \Big(\langle Lx \mid Ly - J_{\gamma B}(Ly) \rangle_{\mathcal{G}} + \langle J_{\gamma B}(Ly) \mid Lx^* \rangle_{\mathcal{G}} - \langle J_{\gamma B}(Ly) \mid Ly - J_{\gamma B}(Ly) \rangle_{\mathcal{G}} \Big) \\
&\leqslant \sup_{v \in \mathcal{G}} \Big(\langle Lx \mid v - J_{\gamma B} v \rangle_{\mathcal{G}} + \langle J_{\gamma B} v \mid Lx^* \rangle_{\mathcal{G}} - \langle J_{\gamma B} v \mid v - J_{\gamma B} v \rangle_{\mathcal{G}} \Big) \\
&= F_{\gamma B}(Lx, Lx^*).
\end{aligned} \tag{4.23}$$

Therefore, taking the supremum over $y \in \mathcal{H}$ in (4.23), the conclusion follows from (4.22).

(ii): By Proposition 3.1(vi), Definition 1.1, (4.21), and (i),

$$F_{\gamma(L \overset{\gamma}{\blacklozenge} B)}(x, x^*) = F_{(L \diamond (\gamma B)^{-1})^{-1}}(x, x^*) = F_{L \diamond (\gamma B)^{-1}}(x^*, x) \leqslant F_{(\gamma B)^{-1}}(Lx^*, Lx) = F_{\gamma B}(Lx, Lx^*), \tag{4.24}$$

which completes the proof. ☐

Corollary 4.17. *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ is an isometry, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, and let $\gamma \in \,]0, +\infty[$. Then*

$$(\forall x \in \mathcal{H})(\forall x^* \in \mathcal{H}) \quad F_{\gamma(L \overset{\gamma}{\diamond} B)}(x, x^*) \leqslant F_{\gamma B}(Lx, Lx^*) \tag{4.25}$$

Proof. Since L is an isometry, $\ker(\mathrm{Id}_{\mathcal{G}} - L^* \circ L) = \mathcal{H}$. Therefore, the conclusion is a consequence of Proposition 4.16(i). ☐

Corollary 4.18 ([6, Theorem 2.13]). *Let $0 \neq p \in \mathbb{N}$ and let $\gamma \in \,]0, +\infty[$. For every $k \in \{1, \ldots, p\}$, let $B_k\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone and let $\alpha_k \in \,]0, +\infty[$. Suppose that $\sum_{k=1}^{p} \alpha_k = 1$. Then*

$$F_{\gamma\, \mathrm{rav}_\gamma(B_k, \alpha_k)_{1 \leqslant k \leqslant p}} \leqslant \sum_{k=1}^{p} \alpha_k F_{\gamma B_k}. \tag{4.26}$$

Proof. Define L and B as in Example 3.11 and recall that $L \overset{\gamma}{\diamond} B = \mathrm{rav}_\gamma(B_k, \alpha_k)_{1 \leqslant k \leqslant p}$. In this case, L is an isometry, and it follows from Corollary 4.17 that, for every $x \in \mathcal{H}$ and $x^* \in \mathcal{H}$,

$$F_{\gamma\, \mathrm{rav}_\gamma(B_k, \alpha_k)_{1 \leqslant k \leqslant p}}(x, x^*) \leqslant F_{\gamma B}(Lx, Lx^*) = \sum_{k=1}^{p} \alpha_k F_{\gamma B_k}(x, x^*), \tag{4.27}$$

as announced. ☐

§5. Asymptotic behavior of resolvent compositions

We examine the convergence of the operators $L \overset{\gamma}{\diamond} B$ and $L \overset{\gamma}{\blacklozenge} B$ when γ varies, studying their corresponding graph. We begin by recalling some definitions related to set-convergence, which enable us to characterize the convergence of operators through their graphs.

5.1. Set-convergence

Definition 5.1 (Painlevé–Kuratowski). Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathcal{H}$. The *lower limit* of the sequence $(C_n)_{n\in\mathbb{N}}$ is the closed subset of $\mathcal{H}$ defined by

$$\underline{\lim}\, C_n = \big\{x \in \mathcal{H} \mid (\exists\, (x_n)_{n\in\mathbb{N}} \text{ in } \mathcal{H})(\forall n \in \mathbb{N})\ x_n \in C_n \text{ and } x_n \to x\big\}. \tag{5.1}$$

The *upper limit* of the sequence $(C_n)_{n\in\mathbb{N}}$ is the closed subset of $\mathcal{H}$ defined by

$$\overline{\lim}\, C_n = \big\{x \in \mathcal{H} \mid (\exists\, (x_n)_{n\in\mathbb{N}} \text{ in } \mathcal{H})(\exists\, (k_n)_{n\in\mathbb{N}} \text{ in } \mathbb{N})(\forall n \in \mathbb{N})\ x_n \in C_{k_n} \text{ and } x_n \to x\big\}. \tag{5.2}$$

The sequence $(C_n)_{n\in\mathbb{N}}$ is *Painlevé–Kuratowski* convergent if its upper limit coincides with its lower limit. The limit set in this case is given by

$$\lim_{n\to+\infty} C_n = \underline{\lim}\, C_n = \overline{\lim}\, C_n. \tag{5.3}$$

Let C and D be subsets of $\mathcal{H}$. The *excess function* of C on D is defined by

$$e(C, D) = \sup_{x\in C} d_D(x), \tag{5.4}$$

with the convention that $e(\varnothing, D) = 0$.

Definition 5.2 (ρ-Hausdorff distance [3, 5]). Let C and D be subsets of $\mathcal{H}$, let $\rho \in [0, +\infty[$, and set $C_\rho = C \cap B(0; \rho)$ and $D_\rho = D \cap B(0; \rho)$ The *ρ-Hausdorff distance* between C and D is

$$\operatorname{haus}_\rho(C, D) = \max\{e(C_\rho, D), e(D_\rho, C)\}. \tag{5.5}$$

A sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of $\mathcal{H}$ converges with respect to the ρ-Hausdorff distance to the subset C of $\mathcal{H}$ if

$$(\forall \rho \in\,]0, +\infty[) \quad \lim_{n\to+\infty} \operatorname{haus}_\rho(C_n, C) = 0. \tag{5.6}$$

5.2. Graph-convergence of operators

Definition 5.3. Let $(A_n)_{n\in\mathbb{N}}$ and A be set-valued operators from $\mathcal{H}$ to $2^{\mathcal{H}}$. Then $(A_n)_{n\in\mathbb{N}}$ *graph-converges* to A, denoted by $A_n \xrightarrow{g} A$, if $(\operatorname{gra} A_n)_{n\in\mathbb{N}}$ converges to $\operatorname{gra} A$ in the Painlevé–Kuratowski sense.

Definition 5.4. Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, $B\colon \mathcal{H} \to 2^{\mathcal{H}}$, and $\rho \in [0, +\infty[$. The *ρ-Hausdorff distance* between A and B is

$$\operatorname{haus}_\rho(A, B) = \operatorname{haus}_\rho(\operatorname{gra} A, \operatorname{gra} B). \tag{5.7}$$

A sequence $(A_n)_{n\in\mathbb{N}}$ of operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ converges with respect to the ρ-Hausdorff distance to the operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ if

$$(\forall \rho \in\,]0, +\infty[) \quad \lim_{n\to+\infty} \operatorname{haus}_\rho(A_n, A) = 0. \tag{5.8}$$

Some equivalences of graph-convergence for maximally monotone operators are summarized in the following result (see e.g. [2]).

Lemma 5.5. *Let $(A_n)_{n\in\mathbb{N}}$ and A be maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$. Then the following are equivalent:*

(i) $A_n \xrightarrow{g} A$.

(ii) $(\forall\gamma\in\,]0,+\infty[)(\forall x\in\mathcal{H})\ J_{\gamma A_n}x \to J_{\gamma A}x$.

(iii) $(\exists\,\gamma\in\,]0,+\infty[)(\forall x\in\mathcal{H})\ J_{\gamma A_n}x \to J_{\gamma A}x$.

Lemma 5.6. *Let $(A_n)_{n\in\mathbb{N}}$ and A be maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$, and let $(\gamma_n)_{n\in\mathbb{N}}$ and γ be in $]0,+\infty[$. Suppose that $A_n \xrightarrow{g} A$ and $\gamma_n \to \gamma$. Then the following hold:*

(i) $A_n^{-1} \xrightarrow{g} A^{-1}$.

(ii) $\gamma_n A_n \xrightarrow{g} \gamma A$

Proof. (i): This follows from (2.3) and Lemma 5.5.

(ii): Let $x\in\mathcal{H}$ and set $(\forall n\in\mathbb{N})\ \theta_n = 1-\gamma_n/\gamma$. By [7, Proposition 23.31(iii)],

$$\begin{aligned}\|J_{\gamma_n A_n}x - J_{\gamma A}x\|_{\mathcal{H}} &\leqslant \|J_{\gamma_n A_n}x - J_{\gamma A_n}x\|_{\mathcal{H}} + \|J_{\gamma A_n}x - J_{\gamma A}x\|_{\mathcal{H}}\\ &\leqslant |\theta_n|\|x - J_{\gamma A_n}x\|_{\mathcal{H}} + \|J_{\gamma A_n}x - J_{\gamma A}x\|_{\mathcal{H}}.\end{aligned}\tag{5.9}$$

Further, $A_n \xrightarrow{g} A$ and Lemma 5.5 yield $J_{\gamma A_n}x \to J_{\gamma A}x$. Altogether, since $\theta_n\to 0$, we deduce from (5.9) that $J_{\gamma_n A_n}x \to J_{\gamma A}x$. Finally, invoking Lemma 5.5 one more, we obtain the assertion. ☐

Lemma 5.7 ([4, Propositions 1.1 and 1.2]). *Let $A_1\colon\mathcal{H}\to 2^{\mathcal{H}}$ and $A_2\colon\mathcal{H}\to 2^{\mathcal{H}}$ be maximally monotone, and let $\gamma\in\,]0,+\infty[$. Consider*

$$(\forall\delta\in[0,+\infty[)\quad d_{\gamma,\delta}(A_1,A_2) = \sup_{x\in B(0;\delta)} \|J_{\gamma A_1}x - J_{\gamma A_2}x\|_{\mathcal{H}}.\tag{5.10}$$

Then the following hold:

(i) $(\forall\rho\in[0,+\infty[)\ \mathrm{haus}_\rho(A_1,A_2) \leqslant \max\{1,\gamma^{-1}\}d_{\gamma,(1+\gamma)\rho}(A_1,A_2)$.

(ii) *Set* $\rho = \max\{\delta + \|J_{\gamma A_1}0\|_{\mathcal{H}}, \gamma^{-1}(\delta + \|J_{\gamma A_1}0\|_{\mathcal{H}}\}$. *Then* $d_{\gamma,\delta}(A_1,A_2) \leqslant (2+\gamma)\,\mathrm{haus}_\rho(A_1,A_2)$.

5.3. Convergence of resolvent compositions

We proceed to study the graph-convergence and convergence with respect to the ρ-Hausdorff distance of resolvent compositions.

Proposition 5.8. *Let $(L_n)_{n\in\mathbb{N}}$ and L be in $\mathcal{B}(\mathcal{H},\mathcal{G})$, let $(B_n)_{n\in\mathbb{N}}$ and B be maximally monotone operators from $\mathcal{G}$ to $2^{\mathcal{G}}$, and let $(\gamma_n)_{n\in\mathbb{N}}$ and γ be in $]0,+\infty[$. Suppose that $L_n\to L$, $B_n \xrightarrow{g} B$, $\gamma_n\to\gamma$, and $(\forall n\in\mathbb{N})$ $\|L_n\|\leqslant 1$. Then the following hold:*

(i) $L_n \overset{\gamma_n}{\diamond} B_n \xrightarrow{g} L \overset{\gamma}{\diamond} B$.

(ii) $L_n \overset{\gamma_n}{\blacklozenge} B_n \xrightarrow{g} L \overset{\gamma}{\blacklozenge} B$.

Proof. We recall from Corollary 4.5(ii) that the operators $(L_n \overset{\gamma_n}{\diamond} B_n)_{n\in\mathbb{N}}$, $L \overset{\gamma}{\diamond} B$, $(L_n \overset{\gamma_n}{\blacklozenge} B_n)_{n\in\mathbb{N}}$, and $L \overset{\gamma}{\blacklozenge} B$ are maximally monotone. Therefore, by Lemma 5.5, it is enough to verify the convergence of the resolvent of these operators.

(i): Let $x \in \mathcal{H}$ and set $(\forall n \in \mathbb{N})$ $\theta_n = 1 - \gamma/\gamma_n$. It follows from [7, Proposition 23.31(iii)], Proposition 3.2(i), and Lemma 5.6 that

$$\begin{aligned}\|J_{\gamma(L_n \overset{\gamma_n}{\diamond} B_n)}x - J_{\gamma(L \overset{\gamma}{\diamond} B)}x\|_{\mathcal{H}} &\leqslant \|J_{\gamma(L_n \overset{\gamma_n}{\diamond} B_n)}x - J_{\gamma_n(L_n \overset{\gamma_n}{\diamond} B_n)}x\|_{\mathcal{H}} + \|J_{\gamma_n(L_n \overset{\gamma_n}{\diamond} B_n)}x - J_{\gamma(L \overset{\gamma}{\diamond} B)}x\|_{\mathcal{H}} \\ &\leqslant |\theta_n|\,\|x - J_{\gamma_n(L_n \overset{\gamma_n}{\diamond} B_n)}x\|_{\mathcal{H}} + \|L_n^*\big(J_{\gamma B_n}(L_n x)\big) - L^*\big(J_{\gamma B}(Lx)\big)\|_{\mathcal{H}} \\ &= |\theta_n|\,\|x - L_n^*\big(J_{\gamma_n B_n}(L_n x)\big)\|_{\mathcal{H}} + \|L_n^*\big(J_{\gamma B_n}(L_n x)\big) - L^*\big(J_{\gamma B}(Lx)\big)\|_{\mathcal{H}}.\end{aligned} \tag{5.11}$$

Further, nonexpansiveness of $J_{\gamma_n B_n}$ implies that

$$\begin{aligned}\|J_{\gamma_n B_n}(L_n x) - J_{\gamma B}(Lx)\|_{\mathcal{G}} &\leqslant \|J_{\gamma_n B_n}(L_n x) - J_{\gamma_n B_n}(Lx)\|_{\mathcal{G}} + \|J_{\gamma_n B_n}(Lx) - J_{\gamma B}(Lx)\|_{\mathcal{G}} \\ &\leqslant \|L_n x - Lx\|_{\mathcal{G}} + \|J_{\gamma_n B_n}(Lx) - J_{\gamma B}(Lx)\|_{\mathcal{G}}.\end{aligned} \tag{5.12}$$

On the other hand, by Lemma 5.6(ii), $\gamma_n B \overset{g}{\to} \gamma B$. Since $\theta_n \to 0$ and $L_n \to L$, we combine (5.11) and (5.12) to obtain $J_{\gamma(L_n \overset{\gamma_n}{\diamond} B_n)}x \to J_{\gamma(L \overset{\gamma}{\diamond} B)}x$. Therefore, the conclusion follows from Lemma 5.5.

(ii): By Lemma 5.6(i), $B_n^{-1} \overset{g}{\to} B^{-1}$. Therefore, (i) yields $L_n \overset{1/\gamma_n}{\diamond} B_n^{-1} \overset{g}{\to} L \overset{1/\gamma}{\diamond} B^{-1}$. Altogether, by Definition 1.1 and Lemma 5.6(i) once more,

$$L_n \overset{\gamma_n}{\blacklozenge} B_n = \big(L_n \overset{1/\gamma_n}{\diamond} B_n^{-1}\big)^{-1} \overset{g}{\to} \big(L \overset{1/\gamma}{\diamond} B^{-1}\big)^{-1} = L \overset{\gamma}{\blacklozenge} B, \tag{5.13}$$

as asserted. ☐

Proposition 5.9. *Let $(L_n)_{n\in\mathbb{N}}$ and L be in $\mathcal{B}(\mathcal{H},\mathcal{G})$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0,+\infty[$. Suppose that $L_n \to L$ and that $(\forall n \in \mathbb{N})$ $\|L_n\| \leqslant 1$. Then the following hold:*

(i) *Suppose that $\gamma_n \downarrow 0$. Then the following are satisfied:*

  (a) $L_n \diamond (\gamma_n B) \overset{g}{\to} L \diamond N_{\overline{\operatorname{dom} B}}$.

  (b) $L_n \blacklozenge (\gamma_n B) \overset{g}{\to} L \blacklozenge N_{\overline{\operatorname{dom} B}}$.

(ii) *Suppose that $\gamma_n \uparrow +\infty$ and that $\operatorname{zer} B \neq \varnothing$. Then the following are satisfied:*

  (a) $L_n \diamond (\gamma_n B) \overset{g}{\to} L \diamond N_{\operatorname{zer} B}$.

  (b) $L_n \blacklozenge (\gamma_n B) \overset{g}{\to} L \blacklozenge N_{\operatorname{zer} B}$.

Proof. (i): Let $y \in \mathcal{G}$. We recall from [7, Corollary 21.14] that $\overline{\operatorname{dom} B}$ is closed and convex. Further, by [7, Theorem 23.48], $J_{\gamma_n B}y \to \operatorname{proj}_{\overline{\operatorname{dom} B}} y = J_{N_{\overline{\operatorname{dom} B}}}y$. Therefore, it follows from Lemma 5.5 that $\gamma_n B \overset{g}{\to} N_{\overline{\operatorname{dom} B}}$, and the conclusion is a consequence of Proposition 5.8.

(ii): Let $y \in \mathcal{G}$. We recall from [7, Proposition 23.39] that $\operatorname{zer} B$ is closed and convex. Further, by [7, Theorem 23.48], $J_{\gamma_n B}y \to \operatorname{proj}_{\operatorname{zer} B} y = J_{N_{\operatorname{zer} B}}y$. Therefore, it follows from Lemma 5.5 that $\gamma_n B \overset{g}{\to} N_{\operatorname{zer} B}$, and the conclusion is a consequence of Proposition 5.8. ☐

The following proposition shows that, for a fixed $\gamma \in]0,+\infty[$, resolvent compositions are nonexpansive with respect to $d_{\gamma,\delta}$.

Proposition 5.10. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $\|L\| \leqslant 1$, let $B_1\colon \mathcal{G} \to 2^{\mathcal{G}}$ and $B_2\colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, let $\gamma \in]0,+\infty[$, and let $\delta \in]0,+\infty[$. Then the following hold:*

(i) $d_{\gamma,\delta}(L \overset{\gamma}{\diamond} B_1, L \overset{\gamma}{\diamond} B_2) = d_{\gamma,\delta}(L \overset{\gamma}{\blacklozenge} B_1, L \overset{\gamma}{\blacklozenge} B_2)$.

(ii) $d_{\gamma,\delta}(L \overset{\gamma}{\diamond} B_1, L \overset{\gamma}{\diamond} B_2) \leqslant \|L\|\, d_{\gamma,\|L\|\delta}(B_1, B_2)$.

(iii) $d_{\gamma,\delta}(L \overset{\gamma}{\blacklozenge} B_1, L \overset{\gamma}{\blacklozenge} B_2) \leqslant \|L\|\, d_{\gamma,\|L\|\delta}(B_1, B_2)$.

Proof. We recall from Corollary 4.5(ii) that, for every $k \in \{1,2\}$, $L \overset{\gamma}{\diamond} B_k$ and $L \overset{\gamma}{\blacklozenge} B_k$ are maximally monotone.

(i): By Proposition 3.2, $J_{\gamma(L \overset{\gamma}{\diamond} B_1)} - J_{\gamma(L \overset{\gamma}{\diamond} B_2)} = J_{\gamma(L \overset{\gamma}{\blacklozenge} B_1)} - J_{\gamma(L \overset{\gamma}{\blacklozenge} B_2)}$. Therefore, the conclusion follows from (5.10).

(ii): Let $x \in B(0;\delta)$. It follows from Proposition 3.2(i) that

$$\begin{aligned}
\|J_{\gamma(L \overset{\gamma}{\diamond} B_1)}x - J_{\gamma(L \overset{\gamma}{\diamond} B_2)}x\|_{\mathcal{H}} &= \|L^*\big(J_{\gamma B_1}(Lx)\big) - L^*\big(J_{\gamma B_2}(Lx)\big)\|_{\mathcal{H}} \\
&\leqslant \|L\|\, \|J_{\gamma B_1}(Lx) - J_{\gamma B_2}(Lx)\|_{\mathcal{G}} \\
&\leqslant \|L\| \sup_{u \in B(0;\|L\|\delta)} \|J_{\gamma B_1}u - J_{\gamma B_2}u\|_{\mathcal{G}} \\
&= \|L\|\, d_{\gamma,\|L\|\delta}(B_1, B_2).
\end{aligned} \tag{5.14}$$

Therefore, by taking the supremum over all $x \in B(0;\delta)$, we obtain the assertion.

(iii): A consequence of (i) and (ii). ☐

Proposition 5.11. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $\|L\| \leqslant 1$, let $B_1 \colon \mathcal{G} \to 2^{\mathcal{G}}$ and $B_2 \colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, let $\gamma \in \left]0,+\infty\right[$, and let $\rho \in \left]0,+\infty\right[$. Then*

$$\operatorname{haus}_\rho(L \overset{\gamma}{\diamond} B_1, L \overset{\gamma}{\diamond} B_2) \leqslant \max\{1, \gamma^{-1}\}\|L\| d_{\gamma,\|L\|(1+\gamma)\rho}(B_1, B_2). \tag{5.15}$$

Proof. Combine Corollary 4.5(ii), Lemma 5.7(i), and Proposition 5.10(ii). ☐

Proposition 5.12. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $\|L\| \leqslant 1$ and let $B \colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone. Assume that $L^* \circ B \circ L$ is maximally monotone. Then the following hold:*

(i) $L \overset{\gamma}{\blacklozenge} B \overset{g}{\to} L^* \circ B \circ L$ *as* $0 < \gamma \to 0$.

(ii) *Suppose that one of the following is satisfied:*

  (a) $\operatorname{ran}(B \circ L)$ *is bounded.*

  (b) *There exists $S \in \mathcal{B}(\mathcal{H},\mathcal{G})$ such that $S \circ L^*$ is invertible.*

  Then

$$(\forall \rho \in \left]0,+\infty\right[) \quad \lim_{\gamma \to 0} \operatorname{haus}_\rho(L \overset{\gamma}{\blacklozenge} B, L^* \circ B \circ L) = 0. \tag{5.16}$$

Proof. Let $\gamma \in \left]0,1\right[$ and recall from Corollary 4.5(ii) that $L \overset{\gamma}{\blacklozenge} B$ is maximally monotone. Let $x \in \mathcal{H}$, let $\rho \in \left]0,+\infty\right[$, and suppose that $x \in B(0;2\rho)$. Set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$, set $p = J_{L^* \circ B \circ L}x$, and set $p_\gamma = J_{L \overset{\gamma}{\blacklozenge} B}x$. We deduce from Proposition 3.3(ii) that

$$\begin{aligned}
x - p_\gamma \in (L \overset{\gamma}{\blacklozenge} B)p_\gamma &\Leftrightarrow x - p_\gamma \in L^*\Big(\big(B^{-1} + \gamma\Psi\big)^{-1}(Lp_\gamma)\Big) \\
&\Leftrightarrow \begin{cases} (\exists\, y_\gamma \in \mathcal{G}) \quad x - p_\gamma = L^* y_\gamma \\ Lp_\gamma \in \big(B^{-1} + \gamma\Psi\big)y_\gamma \end{cases} \\
&\Leftrightarrow \begin{cases} (\exists\, y_\gamma \in \mathcal{G}) \quad x - p_\gamma = L^* y_\gamma \\ y_\gamma \in B(Lp_\gamma - \gamma\Psi y_\gamma). \end{cases}
\end{aligned} \tag{5.17}$$

On the other hand,

$$x - p \in L^*\big(B(Lp)\big) \Leftrightarrow (\exists\, y \in \mathcal{G}) \quad x - p = L^* y \quad \text{and} \quad y \in B(Lp). \tag{5.18}$$

Altogether, monotonicity of B, (5.17), and (5.18) yield

$$\begin{aligned} \langle (Lp_\gamma - \gamma\Psi y_\gamma) - Lp \mid y_\gamma - y\rangle_{\mathcal{G}} \geqslant 0 &\Leftrightarrow \langle p_\gamma - p \mid L^*(y_\gamma - y)\rangle_{\mathcal{H}} - \gamma\langle \Psi y_\gamma \mid y_\gamma - y\rangle_{\mathcal{G}} \geqslant 0 \\ &\Leftrightarrow \langle p_\gamma - p \mid p - p_\gamma\rangle_{\mathcal{H}} - \gamma\langle \Psi y_\gamma \mid y_\gamma - y\rangle_{\mathcal{G}} \geqslant 0 \\ &\Leftrightarrow \|p_\gamma - p\|_{\mathcal{H}}^2 + \gamma\langle \Psi y_\gamma \mid y_\gamma - y\rangle_{\mathcal{G}} \leqslant 0. \end{aligned} \tag{5.19}$$

Further, since $L^* y_\gamma = x - p_\gamma$ and $L^* y = x - p$, by Cauchy–Schwarz inequality [7, Fact 2.11],

$$\begin{aligned} \langle \Psi y_\gamma \mid y_\gamma - y\rangle_{\mathcal{G}} &= \langle y_\gamma - L(x - p_\gamma) \mid y_\gamma - y\rangle_{\mathcal{G}} \\ &= \langle y_\gamma \mid y_\gamma - y\rangle_{\mathcal{G}} - \langle x - p_\gamma \mid L^*(y_\gamma - y)\rangle_{\mathcal{H}} \\ &= \|y_\gamma\|_{\mathcal{G}}^2 - \langle y_\gamma \mid y\rangle_{\mathcal{G}} - \langle x - p_\gamma \mid p - p_\gamma\rangle_{\mathcal{H}} \\ &\geqslant \|y_\gamma\|_{\mathcal{G}}^2 - \|y_\gamma\|_{\mathcal{G}}\,\|y\|_{\mathcal{G}} - \big(\langle p - p_\gamma \mid p - p_\gamma\rangle_{\mathcal{H}} + \langle x - p \mid p - p_\gamma\rangle_{\mathcal{H}}\big) \\ &\geqslant \|y_\gamma\|_{\mathcal{G}}^2 - \|y_\gamma\|_{\mathcal{G}}\,\|y\|_{\mathcal{G}} - \|p - p_\gamma\|_{\mathcal{H}}^2 - \|x - p\|_{\mathcal{H}}\,\|p - p_\gamma\|_{\mathcal{H}} \\ &\geqslant \min_{\alpha\in\mathbb{R}}\big(\alpha^2 - \alpha\|y\|_{\mathcal{G}}\big) - \|p - p_\gamma\|_{\mathcal{H}}^2 - \|x - p\|_{\mathcal{H}}\,\|p - p_\gamma\|_{\mathcal{H}} \\ &= -\frac{1}{4}\|y\|_{\mathcal{G}}^2 - \|p - p_\gamma\|_{\mathcal{H}}^2 - \|x - p\|_{\mathcal{H}}\,\|p - p_\gamma\|_{\mathcal{H}}. \end{aligned} \tag{5.20}$$

Set $\delta = 2\rho + \|J_{L^*\circ B\circ L}0\|_{\mathcal{H}}$. Since $L^* \circ B \circ L$ is maximally monotone, nonexpansiveness of $J_{L^*\circ B\circ L}$ yields

$$\|p\|_{\mathcal{H}} \leqslant \|J_{L^*\circ B\circ L}x - J_{L^*\circ B\circ L}0\|_{\mathcal{H}} + \|J_{L^*\circ B\circ L}0\|_{\mathcal{H}} \leqslant \|x\|_{\mathcal{H}} + \|J_{L^*\circ B\circ L}0\|_{\mathcal{H}} \leqslant 2\rho + \|J_{L^*\circ B\circ L}0\|_{\mathcal{H}} = \delta. \tag{5.21}$$

Thus, we combine (5.19), (5.20), and (5.21) to deduce that

$$\begin{aligned} &\|p - p_\gamma\|_{\mathcal{H}}^2 - \frac{\gamma}{4}\|y\|_{\mathcal{G}}^2 - \gamma\|p - p_\gamma\|_{\mathcal{H}}^2 - \gamma\|x - p\|_{\mathcal{H}}\,\|p - p_\gamma\|_{\mathcal{H}} \leqslant 0 \\ &\Leftrightarrow (1-\gamma)\|p - p_\gamma\|_{\mathcal{H}}^2 - \gamma\|x - p\|_{\mathcal{H}}\,\|p - p_\gamma\|_{\mathcal{H}} - \frac{\gamma}{4}\|y\|_{\mathcal{G}}^2 \leqslant 0 \\ &\Leftrightarrow \|p - p_\gamma\|_{\mathcal{H}}^2 - \frac{\gamma}{1-\gamma}\|x - p\|_{\mathcal{H}}\,\|p - p_\gamma\|_{\mathcal{H}} - \frac{\gamma}{4(1-\gamma)}\|y\|_{\mathcal{G}}^2 \leqslant 0 \\ &\Rightarrow \|p - p_\gamma\|_{\mathcal{H}}^2 - \frac{\gamma}{1-\gamma}(\|x\|_{\mathcal{H}} + \|p\|_{\mathcal{H}})\|p - p_\gamma\|_{\mathcal{H}} - \frac{\gamma}{4(1-\gamma)}\|y\|_{\mathcal{G}}^2 \leqslant 0 \\ &\Rightarrow \|p - p_\gamma\|_{\mathcal{H}}^2 - \frac{\gamma}{1-\gamma}(2\rho + \delta)\|p - p_\gamma\|_{\mathcal{H}} - \frac{\gamma}{4(1-\gamma)}\|y\|_{\mathcal{G}}^2 \leqslant 0 \\ &\Rightarrow \left(\|p - p_\gamma\|_{\mathcal{H}} - \frac{\gamma}{2(1-\gamma)}(2\rho + \delta)\right)^2 \leqslant \frac{\gamma^2}{4(1-\gamma)^2}(2\rho + \delta)^2 + \frac{\gamma}{4(1-\gamma)}\|y\|_{\mathcal{G}}^2 \\ &\Rightarrow \|p - p_\gamma\|_{\mathcal{H}} \leqslant \left(\frac{\gamma^2}{4(1-\gamma)^2}(2\rho + \delta)^2 + \frac{\gamma}{4(1-\gamma)}\|y\|_{\mathcal{G}}^2\right)^{1/2} + \frac{\gamma}{2(1-\gamma)}(2\rho + \delta). \end{aligned} \tag{5.22}$$

(i): By (5.22), $J_{L\overset{\gamma}{\bullet}B}x \to J_{L^*\circ B\circ L}x$ as $0 < \gamma \to 0$, and the conclusion follows from Lemma 5.5.

(ii): Assumption (ii)(a) implies that there exits $\eta \in \left]0, +\infty\right[$ such that, for every $z \in \operatorname{ran}(B \circ L)$, $\|z\|_{\mathcal{G}} \leqslant \eta$. In particular, (5.18) yields $\|y\|_{\mathcal{G}} \leqslant \eta$. On the other hand, Assumption (ii)(b) and (5.18) imply that $y = (S\circ L^*)^{-1}(S(x-p))$. Thus, $\|y\|_{\mathcal{G}} \leqslant \|(S\circ L^*)^{-1}\|\,\|S\|(2\rho+\delta)$. Therefore, either Assumption (ii)(a) or Assumption (ii)(b) implies that there exists $\eta \in]0,+\infty[$ such that $\|y\|_{\mathcal{G}} \leqslant \eta$. Altogether, we deduce from Lemma 5.7(i) and (5.22) that

$$\begin{aligned}
\operatorname{haus}_\rho(L \overset{\gamma}{\blacklozenge} B, L^* \circ B \circ L) &\leqslant d_{1,2\rho}(L \overset{\gamma}{\blacklozenge} B, L^* \circ B \circ L)\\
&= \sup_{u \in B(0;2\rho)} \|J_{L \overset{\gamma}{\blacklozenge} B} u - J_{L^* \circ B \circ L} u\|_{\mathcal{H}}\\
&\leqslant \left(\frac{\gamma^2}{4(1-\gamma)^2}(2\rho+\delta)^2 + \frac{\gamma}{4(1-\gamma)}\eta^2\right)^{1/2} + \frac{\gamma}{2(1-\gamma)}(2\rho+\delta)\\
&\to 0 \quad \text{as} \quad 0 < \gamma \to 0,
\end{aligned} \tag{5.23}$$

which completes the proof. ☐

Corollary 5.13. *Let $0 \neq p \in \mathbb{N}$ and, for every $k \in \{1,\ldots,p\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $L_k \in \mathcal{B}(\mathcal{H},\mathcal{G}_k)$, let $B_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ be maximally monotone, and let $\alpha_k \in]0,+\infty[$. Suppose that $\sum_{k=1}^p \alpha_k \|L_k\|^2 \leqslant 1$ and that $\sum_{k=1}^p \alpha_k L_k^* \circ B_k \circ L_k$ is maximally monotone. Then*

$$\overset{\blacklozenge}{\mathrm{M}}_\gamma(B_k, L_k)_{1 \leqslant k \leqslant p} \overset{g}{\to} \sum_{k=1}^p \alpha_k L_k^* \circ B_k \circ L_k \quad \textit{as} \quad 0 < \gamma \to 0. \tag{5.24}$$

Proof. Define L as in (3.20) and B as in (3.21), and recall that from Example 4.6 that, for every $\gamma \in]0,+\infty[$, $\overset{\blacklozenge}{\mathrm{M}}_\gamma(B_k, \alpha_k)_{1 \leqslant k \leqslant p} = L \overset{\gamma}{\blacklozenge} B$ is maximally monotone. Therefore, since $\sum_{k=1}^p \alpha_k L_k^* \circ B_k \circ L_k = L^* \circ B \circ L$, the conclusion follows from Proposition 5.12(i). ☐

Corollary 5.14 ([4, Proposition 1.4]). *Let $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone. Then*

$$(\forall \rho \in]0,+\infty[) \quad \lim_{\gamma \to 0} \operatorname{haus}_\rho({}^{\gamma}\!A, A) = 0. \tag{5.25}$$

Proof. Set $L = \mathrm{Id}_{\mathcal{H}}/2$ and $B = 2A(2\mathrm{Id}_{\mathcal{H}})$. Thus, $L^* \circ B \circ L = A$. Further, by Example 3.6, $(\forall \gamma \in]0,+\infty[)$ ${}^{\gamma}\!A = L \overset{\gamma/3}{\blacklozenge} B$. Since $\mathrm{Id}_{\mathcal{H}} \circ L^*$ is invertible, we derive from Proposition 5.12(ii)(b) that

$$(\forall \rho \in]0,+\infty[) \quad \operatorname{haus}_\rho({}^{\gamma}\!A, A) = \operatorname{haus}_\rho(L \overset{\gamma/3}{\blacklozenge} B, L^* \circ B \circ L) \to 0 \quad \text{as} \quad 0 < \gamma \to 0, \tag{5.26}$$

which establishes (5.25). ☐

Corollary 5.15. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$ and let $g \in \Gamma_0(\mathcal{G})$. Assume that $0 \in \operatorname{sri}(\operatorname{dom} g - \operatorname{ran} L)$. Then the following hold:*

(i) $\partial(L \overset{\gamma}{\blacklozenge} g) \overset{g}{\to} \partial(g \circ L)$ *as* $0 < \gamma \to 0$.

(ii) *Suppose that $g \colon \mathcal{G} \to \mathbb{R}$ is β-Lipschitzian for some $\beta \in]0,+\infty[$. Then*

$$(\forall \rho \in]0,+\infty[) \quad \lim_{\gamma \to 0} \operatorname{haus}_\rho\big(\partial(L \overset{\gamma}{\blacklozenge} g), \partial(g \circ L)\big) = 0. \tag{5.27}$$

Proof. Invoking [7, Corollaries 13.38 and 16.53(i)], $g^{**} = g$ and $\partial(g \circ L) = L^* \circ (\partial g) \circ L$.

(i): It follows from Example 4.14(ii) and Proposition 5.12(i) that

$$\partial(L \overset{\gamma}{\blacklozenge} g) = L \overset{\gamma}{\blacklozenge} \partial g \overset{g}{\to} L^* \circ (\partial g) \circ L = \partial(g \circ L) \quad \text{as} \quad 0 < \gamma \to 0. \tag{5.28}$$

(ii): Appealing to [7, Corollary 17.19], $\operatorname{ran} \partial g \subset B(0;\beta)$. Thus, $\operatorname{ran} \partial g$ is bounded, and the conclusion follows from Proposition 5.12(ii)(a). ☐

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