Resolvent Compositions for Positive Linear Operators [*]

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Abstract Resolvent compositions were recently introduced as monotonicity-preserving operations that combine a set-valued monotone operator and a bounded linear operator. They generalize in particular the notion of a resolvent average. We analyze the resolvent compositions when the monotone operator is a positive linear operator. We establish several new properties, including Löwner partial order relations, concavity, and asymptotic behavior. In addition, we show that the resolvent composition operations are nonexpansive with respect to the Thompson metric. We also introduce a new form of geometric interpolation and explore its connections to resolvent compositions. Finally, we study two nonlinear equations based on resolvent compositions.

Keywords. parallel composition, proximal composition, resolvent average, resolvent composition, resolvent mixture, Thompson metric

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§1. Introduction

Throughout, $\mathcal{H}$ is a real Hilbert space with identity operator $\mathrm{Id}_{\mathcal{H}}$, scalar product $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$, and associated norm $\| \cdot \|_{\mathcal{H}}$. In addition, $\mathcal{G}$ is a real Hilbert space, the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$ is denoted by $\mathcal{B}\,(\mathcal{H}, \mathcal{G})$, and $\mathcal{B}\,(\mathcal{H}) = \mathcal{B}\,(\mathcal{H}, \mathcal{H})$. The adjoint of $L \in \mathcal{B}\,(\mathcal{H}, \mathcal{G})$ is denoted by L^*. The set $\mathcal{P}(\mathcal{H})$ of positive operators on $\mathcal{H}$ is the collection of self-adjoint operators $A \in \mathcal{B}\,(\mathcal{H})$ such that $(\forall x \in \mathcal{H})\ \langle Ax \mid x \rangle \geqslant 0$. The Löwner partial ordering between two self-adjoint operators A and B in $\mathcal{B}\,(\mathcal{H})$ is defined by $A \preccurlyeq B \Leftrightarrow B - A \in \mathcal{P}(\mathcal{H})$, and the set of self-adjoint strongly positive operators on $\mathcal{H}$ is

$$\mathcal{S}(\mathcal{H}) = \big\{ A \in \mathcal{P}(\mathcal{H}) \mid (\exists\, \alpha \in \left]0, +\infty\right[)\ \ \alpha \mathrm{Id}_{\mathcal{H}} \preccurlyeq A \big\}. \tag{1.1}$$

A fundamental operator associated with a monotone operator $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ is its resolvent

$$J_B = (\mathrm{Id}_{\mathcal{G}} + B)^{-1}, \tag{1.2}$$

which plays a central role in monotone operator theory and convex optimization, especially through its use in operator splitting algorithms [3, 7, 10]. In many applications, monotone operators arise in combination with linear operators, which motivates the study of operations that combine the monotone operator B and a linear operator $L \in \mathcal{B}\,(\mathcal{H}, \mathcal{G})$ while preserving monotonicity. Recently, [6] introduced two monotonicity-preserving operations called the *resolvent composition* and the *resolvent cocomposition* of B and L, defined respectively by

$$L \overset{\gamma}{\diamond} B = L^* \triangleright \big(B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} \tag{1.3}$$

and

$$L \overset{\gamma}{\blacklozenge} B = \Big(L \overset{1/\gamma}{\diamond} B^{-1}\Big)^{-1}, \tag{1.4}$$

where $L^* \triangleright B = (L^* \circ B^{-1} \circ L)^{-1}$ is the *parallel composition* of B by L^* [3], and $\gamma \in \left]0, +\infty\right[$. An attractive property of resolvent compositions is that their resolvent operators can be explicitly expressed through L and the resolvent of B [6, Propositions 1.2 and 4.1(v)], namely,

$$J_{\gamma(L \overset{\gamma}{\diamond} B)} = L^* \circ J_{\gamma B} \circ L \quad \text{and} \quad J_{\gamma(L \overset{\gamma}{\blacklozenge} B)} = \mathrm{Id}_{\mathcal{H}} - L^* \circ \big(\mathrm{Id}_{\mathcal{G}} - J_{\gamma B}\big) \circ L. \tag{1.5}$$

This feature, in turn, significantly facilitates the design and implementation of algorithms for monotone inclusion and convex optimization problems [5, 6, 7, 8, 12]. Special cases can also be implicitly found in concrete applications such as image recovery [8], neural networks [14], inverse problems [15], and machine learning [29, 32]. For further motivation, let us consider the following examples.

Example 1.1 (resolvent mixtures). Let $0 \neq p \in \mathbb{N}$ and let $\gamma \in \left]0, +\infty\right[$. For every $k \in \{1, \ldots, p\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $L_k \in \mathcal{B}\,(\mathcal{H}, \mathcal{G}_k)$ be such that $0 < \|L_k\| \leqslant 1$, let $B_k \in \mathcal{S}(\mathcal{G}_k)$, and let $\alpha_k \in \left]0, +\infty\right[$. Suppose that $\sum_{k=1}^{p} \alpha_k = 1$. Then the *resolvent mixture* and the *resolvent comixture* are defined, respectively, by

$$\overset{\diamond}{\mathrm{M}}_{\gamma}(L_k, B_k)_{1 \leqslant k \leqslant p} = \left(\sum_{k=1}^{p} \alpha_k L_k^* \circ \big(B_k + \gamma^{-1}\mathrm{Id}_{\mathcal{G}_k}\big)^{-1} \circ L_k \right)^{-1} - \gamma^{-1}\mathrm{Id}_{\mathcal{H}} \tag{1.6}$$

and

$$\overset{\blacklozenge}{\mathrm{M}}_{\gamma}(L_k,B_k)_{1\leqslant k\leqslant p}=\Biggl(\biggl(\sum_{k=1}^{p}\alpha_k L_k^*\circ\bigl(B_k^{-1}+\gamma^{-1}\mathrm{Id}_{\mathcal{G}_k}\bigr)^{-1}\circ L_k\biggr)^{-1}-\gamma^{-1}\mathrm{Id}_{\mathcal{H}}\Biggr)^{-1} \tag{1.7}$$

Resolvent mixtures were introduced in [6, Example 3.4], and subsequently studied in [5, 12]. They are a particular case of resolvent compositions. Specifically, let $\mathcal{G}=\bigoplus_{k=1}^{p}\mathcal{G}_k$ and set

$$L\colon\mathcal{H}\to\mathcal{G}\colon x\mapsto\bigl(\sqrt{\alpha_k}L_kx\bigr)_{1\leqslant k\leqslant p}\quad\text{and}\quad B\colon\mathcal{G}\to\mathcal{G}\colon(y_k)_{1\leqslant k\leqslant p}\mapsto(B_ky_k)_{1\leqslant k\leqslant p}. \tag{1.8}$$

Then $L\overset{\gamma}{\diamond}B=\overset{\diamond}{\mathrm{M}}_{\gamma}(L_k,B_k)_{1\leqslant k\leqslant p}$ and $L\overset{\gamma}{\blacklozenge}B=\overset{\blacklozenge}{\mathrm{M}}_{\gamma}(L_k,B_k)_{1\leqslant k\leqslant p}$. Further, as shown in [12, Proposition 5.13(i)], $\overset{\blacklozenge}{\mathrm{M}}_{\gamma}(L_k,B_k)_{1\leqslant k\leqslant p}$ graph converges to $\sum_{k=1}^{p}\alpha_kL_k^*\circ B_k\circ L_k$ as $\gamma\to 0$.

Example 1.2 (arithmetic, harmonic, and resolvent average). In the context of Example 1.1, suppose that, for every $k\in\{1,\ldots,p\}$, $\mathcal{G}_k=\mathcal{H}$ and $L_k=\mathrm{Id}_{\mathcal{H}}$. Then, the *arithmetic average* and the *harmonic average* are given, respectively, by

$$L^*\circ B\circ L=\sum_{k=1}^{p}\alpha_kB_k\quad\text{and}\quad L^*\blacktriangleright B=\biggl(\sum_{k=1}^{p}\alpha_kB_k^{-1}\biggr)^{-1}. \tag{1.9}$$

An alternative averaging operation is the *resolvent average*, introduced in [4] and further studied in [2, 6, 31], given by

$$\mathrm{rav}_{\gamma}(B_k)_{1\leqslant k\leqslant p}=\Biggl(\sum_{k=1}^{p}\alpha_k\Bigl(B_k+\gamma^{-1}\mathrm{Id}_{\mathcal{H}}\Bigr)^{-1}\Biggr)^{-1}-\gamma^{-1}\mathrm{Id}_{\mathcal{H}},\quad\text{where}\quad\gamma\in\left]0,+\infty\right[. \tag{1.10}$$

The resolvent average is as a special case of the resolvent mixtures [6, Example 1.3], to wit,

$$\overset{\blacklozenge}{\mathrm{M}}_{\gamma}(\mathrm{Id}_{\mathcal{H}},B_k)_{1\leqslant k\leqslant p}=\overset{\diamond}{\mathrm{M}}_{\gamma}(\mathrm{Id}_{\mathcal{H}},B_k)_{1\leqslant k\leqslant p}=\mathrm{rav}_{\gamma}(B_k)_{1\leqslant k\leqslant p}. \tag{1.11}$$

It was established in [19, 22] that, when $\mathcal{S}(\mathcal{H})$ is endowed with the *Thompson metric* $d_T^{\mathcal{H}}$ (see (4.1)), the averages (1.9) and (1.10) are nonexpansive. This property is important as it ensures stability of the averaging processes and is particularly useful in the study of nonlinear equations [19]. Further, in the finite-dimensional setting, [4, Corollary 4.6] shows that the resolvent average is concave, and [4, Theorem 4.2] establishes, by means of a pointwise convergence proof, that the resolvent average interpolates between the arithmetic average ($0<\gamma\to 0$) and the harmonic average ($\gamma\to+\infty$).

Example 1.3 (weighted $\mathscr{A}\#\mathscr{H}$−means). In the finite-dimensional setting of Example 1.2, a family $\bigl(\mathcal{L}_{\gamma}(B_k)_{1\leqslant k\leqslant p}\bigr)_{\gamma\in\mathbb{R}}$ of means interpolating between the arithmetic average and the harmonic average was introduced in [17] (see also [16]). These means, referred to as *weighted $\mathscr{A}\#\mathscr{H}$−means*, are closely related to resolvent averages [17, Proposition 3.5] through the ordering

$$\mathrm{rav}_{\gamma}(B_k)_{1\leqslant k\leqslant p}\preccurlyeq\mathcal{L}_{1/\gamma}(B_k)_{1\leqslant k\leqslant p}\preccurlyeq\sum_{k=1}^{p}\alpha_kB_k, \tag{1.12}$$

and themselves interpolate between the arithmetic average ($\gamma\to+\infty$) and the harmonic average ($\gamma\to-\infty$) [17, Proposition 3.4].

The aim of this paper is to investigate the operations (1.3) and (1.4) when $B \in \mathcal{S}(\mathcal{G})$. We establish several new properties, including Löwner partial order relations, concavity, nonexpansiveness, and asymptotic behavior. This specific setting leads to new results that, in particular, generalize the corresponding asymptotic properties in [12], as well as those of the proximal average established in [4, 17, 19].

The remainder of the paper is organized as follows. In Section 2, we provide our notation and necessary mathematical background. In Section 3, we present several new properties of $(L \overset{\gamma}{\blacklozenge} B)_{\gamma \in]0,+\infty[}$ and $(L \overset{\gamma}{\diamond} B)_{\gamma \in]0,+\infty[}$. In particular, these operations are concave and

- $L \overset{\gamma}{\blacklozenge} B \preccurlyeq L^* \circ B \circ L \quad$ and $\quad L \overset{\gamma}{\blacklozenge} B \to L^* \circ B \circ L\,$ as $\,0 < \gamma \to 0$,
- $L^* \blacktriangleright B \preccurlyeq L \overset{\gamma}{\diamond} B \quad$ and $\quad L \overset{\gamma}{\diamond} B \to L^* \blacktriangleright B\,$ as $\,\gamma \to +\infty$.

In Section 4, we show that the resolvent compositions are nonexpansive with respect to the Thompson metric, in the sense that, for every $A \in \mathcal{S}(\mathcal{G})$ and $B \in \mathcal{S}(\mathcal{G})$,

$$d_T^{\mathcal{H}}\big(L \overset{\gamma}{\blacklozenge} A, L \overset{\gamma}{\blacklozenge} B\big) \leqslant d_T^{\mathcal{G}}(A,B) \quad \text{and} \quad d_T^{\mathcal{H}}\big(L \overset{\gamma}{\diamond} A, L \overset{\gamma}{\diamond} B\big) \leqslant d_T^{\mathcal{G}}(A,B). \tag{1.13}$$

Finally, in Section 5, we introduce the geometric interpolation $\mathcal{L}_\gamma(L,B)$ (see (5.3)) between $L^* \blacktriangleright B$ and $L^* \circ B \circ L$ when L is an isometry, which generalizes the weighted $\mathscr{A}\#\mathscr{H}$–means. We establish the partial order relations

$$L^* \blacktriangleright B \preccurlyeq \mathcal{L}_{-\gamma}(L,B) \preccurlyeq L \overset{\gamma}{\blacklozenge} B \preccurlyeq \mathcal{L}_{1/\gamma}(L,B) \preccurlyeq L^* \circ B \circ L, \tag{1.14}$$

and conclude by studying two nonlinear equations involving resolvent compositions.

§2. Notation and background

The space $\mathcal{B}\,(\mathcal{H},\mathcal{G})$ is endowed with the topology induced by the operator norm

$$\big(\forall L \in \mathcal{B}\,(\mathcal{H},\mathcal{G})\big) \quad \|L\| = \sup_{\substack{x \in \mathcal{H} \\ \|x\|_{\mathcal{H}} \leqslant 1}} \|Lx\|_{\mathcal{G}}. \tag{2.1}$$

Let $L \in \mathcal{B}\,(\mathcal{H},\mathcal{G})$. Then L is an isometry if $L^* \circ L = \mathrm{Id}_{\mathcal{H}}$. Further, L is bounded below if there exists $\alpha \in]0,+\infty[$ such that $(\forall x \in \mathcal{H})\ \alpha\|x\|_{\mathcal{H}} \leqslant \|Lx\|_{\mathcal{G}}$. Equivalently, by [3, Fact 2.26], L is bounded below if and only if L is injective with closed range. In particular, when $\mathcal{H}$ and $\mathcal{G}$ are finite-dimensional, L is bounded below if and only if L is injective.

The quadratic kernel of $A \in \mathcal{P}(\mathcal{H})$ is $\mathcal{Q}_A \colon \mathcal{H} \to \mathbb{R} \colon x \mapsto (1/2)\langle x \mid Ax\rangle_{\mathcal{H}}$. The Legendre conjugate of $f \colon \mathcal{H} \to [-\infty,+\infty]$ is the function

$$f^* \colon \mathcal{H} \to [-\infty,+\infty] \colon x^* \mapsto \sup_{x \in \mathcal{H}} \big(\langle x \mid x^*\rangle_{\mathcal{H}} - f(x)\big), \tag{2.2}$$

and the Moreau envelope of $f \colon \mathcal{H} \to [-\infty,+\infty]$ of parameter $\gamma \in]0,+\infty[$ is

$${}^{\gamma}f \colon \mathcal{H} \to [-\infty,+\infty] \colon x \mapsto \inf_{z \in \mathcal{H}} \bigg(f(z) + \frac{1}{2\gamma}\|x - z\|_{\mathcal{H}}^2\bigg). \tag{2.3}$$

The set of proper lower semicontinuous convex functions from $\mathcal{H}$ to $]-\infty,+\infty]$ is denoted by $\Gamma_0(\mathcal{H})$. Let $L \in \mathcal{B}\,(\mathcal{H},\mathcal{G})$ and $h \colon \mathcal{G} \to [-\infty,+\infty]$. The infimal postcomposition of h by L^* is

$$L^* \blacktriangleright h \colon \mathcal{H} \to [-\infty,+\infty] \colon x \mapsto \inf_{\substack{y \in \mathcal{G} \\ L^* y = x}} h(y), \tag{2.4}$$

the proximal composition of h and L with parameter $\gamma \in \left]0,+\infty\right[$ (see [6, 9]) is

$$L \overset{\gamma}{\diamond} h = \Big(\tfrac{1}{\gamma}(h^*) \circ L\Big)^* - \frac{1}{2\gamma}\|\cdot\|^2_{\mathcal{H}}, \tag{2.5}$$

and the proximal cocomposition of h and L with parameter $\gamma \in \left]0,+\infty\right[$ is

$$L \overset{\gamma}{\blacklozenge} h = \big(L \overset{1/\gamma}{\diamond} h^*\big)^*. \tag{2.6}$$

The following facts will be used subsequently.

Lemma 2.1. *The following properties are satisfied:*

(i) *Let $A \in \mathcal{S}(\mathcal{G})$. Then $\mathcal{Q}_A^* = \mathcal{Q}_{A^{-1}}$.*

(ii) *Let $A \in \mathcal{S}(\mathcal{G})$ and $B \in \mathcal{S}(\mathcal{G})$. Then $A \preccurlyeq B \Leftrightarrow B^{-1} \preccurlyeq A^{-1}$.*

(iii) *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, $A \in \mathcal{P}(\mathcal{G})$, and $B \in \mathcal{P}(\mathcal{G})$. Then $A \preccurlyeq B \Rightarrow L^* \circ A \circ L \preccurlyeq L^* \circ B \circ L$.*

(iv) *Let $A \in \mathcal{P}(\mathcal{G})$ and $B \in \mathcal{P}(\mathcal{G})$. Then $A \preccurlyeq B \Rightarrow \|A\| \leqslant \|B\|$.*

(v) *Let $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, A, and B be self-adjoint operators in $\mathcal{B}(\mathcal{G})$ such that $A_n \to A$, $B_n \to B$, and $(\forall n \in \mathbb{N})$ $A_n \preccurlyeq B_n$. Then $A \preccurlyeq B$.*

Proof. (i)–(ii): See the proof of [3, Example 13.18(i)].

(iii): Let $x \in \mathcal{H}$. Since $A \preccurlyeq B$, $\langle x \mid L^*(A(Lx))\rangle = \langle Lx \mid A(Lx)\rangle \leqslant \langle Lx \mid B(Lx)\rangle = \langle x \mid L^*(B(Lx))\rangle$.

(iv): Since A and B are self-adjoint and $0 \preccurlyeq A \preccurlyeq B$, we deduce from [3, Fact 2.25(iii)] that

$$\|A\| = \sup_{\substack{x\in\mathcal{G}\\ \|x\|_{\mathcal{G}}\leqslant 1}} |\langle Ax \mid x\rangle_{\mathcal{G}}| = \sup_{\substack{x\in\mathcal{G}\\ \|x\|_{\mathcal{G}}\leqslant 1}} \langle Ax \mid x\rangle_{\mathcal{G}} \leqslant \sup_{\substack{x\in\mathcal{G}\\ \|x\|_{\mathcal{G}}\leqslant 1}} \langle Bx \mid x\rangle_{\mathcal{G}} = \sup_{\substack{x\in\mathcal{G}\\ \|x\|_{\mathcal{G}}\leqslant 1}} |\langle Bx \mid x\rangle_{\mathcal{G}}| = \|B\|. \tag{2.7}$$

(v): Since $A_n \to A$ and $B_n \to B$, convergence is in particular pointwise. Thus, for every $x \in \mathcal{H}$, $0 \leqslant \langle x \mid (B_n - A_n)x\rangle \to \langle x \mid (B - A)x\rangle$. Hence, $0 \preccurlyeq B - A$ or, equivalently, $A \preccurlyeq B$. ☐

Lemma 2.2. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $g \in \Gamma_0(\mathcal{G})$, and let $\gamma \in \left]0,+\infty\right[$. Then the following hold:*

(i) $L \overset{\gamma}{\diamond} g = (L \overset{1/\gamma}{\blacklozenge} g^*)^*$.

(ii) $L \overset{\gamma}{\blacklozenge} g \leqslant \min\{L \overset{\gamma}{\diamond} g,\, g \circ L\}$.

(iii) *Set $\Phi = (1/2)\|\cdot\|^2_{\mathcal{G}} - (1/2)\|\cdot\|^2_{\mathcal{H}} \circ L^*$. Then $L \overset{\gamma}{\blacklozenge} g = (g^* + \gamma\Phi)^* \circ L$.*

(iv) *Set $\Phi = (1/2)\|\cdot\|^2_{\mathcal{G}} - (1/2)\|\cdot\|^2_{\mathcal{H}} \circ L^*$. Then $L \overset{\gamma}{\diamond} g = L^* \triangleright (g + \Phi/\gamma)$.*

Proof. Recall that $g = g^{**}$ [3, Corollary 13.38].

(i): [9, Proposition 3.7(iii)].

(ii): [9, Proposition 3.20(ii)–(iii)].

(iii)–(iv): [9, Proposition 3.2(i)–(ii)]. ☐

Lemma 2.3. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ and $B \in \mathcal{P}(\mathcal{G})$. Then the following hold:*

(i) *$L^* \circ B \circ L \in \mathcal{P}(\mathcal{H})$.*

(ii) *$\mathcal{Q}_B \circ L = \mathcal{Q}_{L^*\circ B\circ L}$.*

(iii) *Suppose that $B \in \mathcal{S}(\mathcal{G})$ and that L is bounded below. Then $L^* \circ B \circ L \in \mathcal{S}(\mathcal{H})$ and $L^* \triangleright B \in \mathcal{S}(\mathcal{H})$.*

Proof. (i): Take $A = 0$ in Lemma 2.1(iii).

(ii): For every $x \in \mathcal{H}$, $\mathscr{Q}_B(Lx) = (1/2)\langle Lx \mid B(Lx)\rangle = (1/2)\langle x \mid L^*(B(Lx))\rangle = \mathscr{Q}_{L^*\circ B\circ L}(x)$.

(iii): Since $B \in \mathcal{S}(\mathcal{G})$, there exists $\alpha \in \left]0,+\infty\right[$ such that $\alpha\mathrm{Id}_{\mathcal{G}} \preccurlyeq B$. On the other hand, since L is bounded below, there exists $\beta \in \left]0,+\infty\right[$ such that $\beta^2\mathrm{Id}_{\mathcal{H}} \preccurlyeq L^* \circ L$. Therefore, Lemma 2.1(iii) yields

$$(\alpha\beta^2)\mathrm{Id}_{\mathcal{H}} \preccurlyeq \alpha(L^* \circ L) = L^* \circ (\alpha\mathrm{Id}_{\mathcal{G}}) \circ L \preccurlyeq L^* \circ B \circ L, \tag{2.8}$$

i.e., $L^*\circ B\circ L \in \mathcal{S}(\mathcal{H})$. Similarly, $L^*\circ B^{-1}\circ L \in \mathcal{S}(\mathcal{H})$, which implies that $L^* \triangleright B = (L^*\circ B^{-1}\circ L)^{-1} \in \mathcal{S}(\mathcal{H})$. $\square$

Lemma 2.4 ([12, Proposition 3.3(ii)]). *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, let $\gamma \in \left]0,+\infty\right[$, and set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$. Then $L \overset{\gamma}{\blacklozenge} B = L^* \circ (B^{-1} + \gamma\Psi)^{-1} \circ L$.*

Lemma 2.5 ([12, Proposition 3.4(i)]). *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is an isometry, let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$, and let $\gamma \in \left]0,+\infty\right[$. Then $L \overset{\gamma}{\diamond} B = L \overset{\gamma}{\blacklozenge} B$.*

§3. Resolvent compositions

In this section, we study the resolvent cocomposition operators when $B \in \mathcal{S}(\mathcal{G})$. We strengthen several results obtained in [12], as well as those established specifically for the resolvent average in [4]. The results obtained include comparisons among the different composite operations, as well as an analysis of the asymptotic behavior of $(L \overset{\gamma}{\blacklozenge} B)_{\gamma\in\left]0,+\infty\right[}$ and $(L \overset{\gamma}{\diamond} B)_{\gamma\in\left]0,+\infty\right[}$, as the parameter γ varies.

Proposition 3.1. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $B \in \mathcal{S}(\mathcal{G})$, and let $\gamma \in \left]0,+\infty\right[$. Then the following hold:*

(i) $L \overset{\gamma}{\blacklozenge} B \in \mathcal{P}(\mathcal{H})$.

(ii) $L \overset{\gamma}{\blacklozenge} \mathscr{Q}_B = \mathscr{Q}_{L \overset{\gamma}{\blacklozenge} B}$.

(iii) *Let $\lambda \in \left]0,1\right[$. Then $T_\gamma\colon \mathcal{S}(\mathcal{G}) \to \mathcal{P}(\mathcal{H})\colon A \mapsto L \overset{\gamma}{\blacklozenge} A$ is concave in the sense that*

$$(\forall A \in \mathcal{S}(\mathcal{G}))\quad \lambda\big(L \overset{\gamma}{\blacklozenge} A\big) + (1-\lambda)\big(L \overset{\gamma}{\blacklozenge} B\big) \preccurlyeq L \overset{\gamma}{\blacklozenge} \big(\lambda A + (1-\lambda)B\big). \tag{3.1}$$

(iv) *Suppose that L is bounded below. Then the following are satisfied:*

  (a) $L \overset{\gamma}{\blacklozenge} B \in \mathcal{S}(\mathcal{H})$ *and* $L \overset{\gamma}{\diamond} B \in \mathcal{S}(\mathcal{H})$.

  (b) $L \overset{\gamma}{\diamond} \mathscr{Q}_B = \mathscr{Q}_{L \overset{\gamma}{\diamond} B}$.

  (c) *Let $\lambda \in \left]0,1\right[$. Then $R_\gamma\colon \mathcal{S}(\mathcal{G}) \to \mathcal{S}(\mathcal{H})\colon A \mapsto L \overset{\gamma}{\diamond} A$ is concave in the sense that*

$$(\forall A \in \mathcal{S}(\mathcal{G}))\quad \lambda\big(L \overset{\gamma}{\diamond} A\big) + (1-\lambda)\big(L \overset{\gamma}{\diamond} B\big) \preccurlyeq L \overset{\gamma}{\diamond} \big(\lambda A + (1-\lambda)B\big). \tag{3.2}$$

Proof. Set $\Psi = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$. Since $\|L\| \leqslant 1$, $\Psi \in \mathcal{P}(\mathcal{G})$, which yields $B^{-1} + \gamma\Psi \in \mathcal{S}(\mathcal{G})$. On the other hand, recall from Lemma 2.4 that

$$L \overset{\gamma}{\blacklozenge} B = L^* \circ \big(B^{-1} + \gamma\Psi\big)^{-1} \circ L. \tag{3.3}$$

(i): This follows from (3.3) and Lemma 2.3(i).

(ii): Set $\varPhi = (1/2)\|\cdot\|_{\mathcal{G}}^2 - (1/2)\|\cdot\|_{\mathcal{H}}^2 \circ L^*$ and note that $\varPhi = \mathscr{Q}_{\varPsi}$. It follows from Lemma 2.2(iii), Lemma 2.1(i), Lemma 2.3(ii), and (3.3) that

$$\begin{aligned} L \overset{\gamma}{\blacklozenge} \mathscr{Q}_B &= \big(\mathscr{Q}_B^* + \gamma\varPhi\big)^* \circ L \\ &= \big(\mathscr{Q}_{B^{-1}} + \gamma\mathscr{Q}_{\varPsi}\big)^* \circ L \\ &= \mathscr{Q}^*_{B^{-1}+\gamma\varPsi} \circ L \\ &= \mathscr{Q}_{L^*\circ(B^{-1}+\gamma\varPsi)^{-1}\circ L} \\ &= \mathscr{Q}_{L \overset{\gamma}{\blacklozenge} B}. \end{aligned} \tag{3.4}$$

(iii): By (i), T_γ is well defined. Further, for every $A \in \mathscr{S}(\mathcal{G})$,

$$\begin{aligned} &\lambda\big(L \overset{\gamma}{\blacklozenge} A\big) + (1-\lambda)\big(L \overset{\gamma}{\blacklozenge} B\big) \preccurlyeq L \overset{\gamma}{\blacklozenge} \big(\lambda A + (1-\lambda)B\big) \\ &\Leftrightarrow\ (\forall x \in \mathcal{H})\ \lambda \left\langle \big(L \overset{\gamma}{\blacklozenge} A\big)x \,\middle|\, x\right\rangle_{\mathcal{H}} + (1-\lambda)\left\langle \big(L \overset{\gamma}{\blacklozenge} B\big)x \,\middle|\, x\right\rangle_{\mathcal{H}} \leqslant \left\langle \Big(L \overset{\gamma}{\blacklozenge} \big(\lambda A + (1-\lambda)B\big)\Big)x \,\middle|\, x\right\rangle_{\mathcal{H}} \\ &\Leftrightarrow\ (\forall x \in \mathcal{H})\ \lambda\mathscr{Q}_{L \overset{\gamma}{\blacklozenge} A}(x) + (1-\lambda)\mathscr{Q}_{L \overset{\gamma}{\blacklozenge} B}(x) \leqslant \mathscr{Q}_{\lambda(L \overset{\gamma}{\blacklozenge} A)+(1-\lambda)(L \overset{\gamma}{\blacklozenge} B)}(x). \end{aligned} \tag{3.5}$$

Therefore, it is enough to prove that, for every $x \in \mathcal{H}$, the function $\mathscr{S}(\mathcal{G}) \to \mathbb{R}\colon A \mapsto \mathscr{Q}_{L \overset{\gamma}{\blacklozenge} A}(x)$ is concave. Set $\varPhi = (1/2)\|\cdot\|_{\mathcal{G}}^2 - (1/2)\|\cdot\|_{\mathcal{H}}^2 \circ L^*$. Because $\operatorname{dom}\varPhi = \mathcal{G}$, the identity $(\gamma\varPhi)^* = \varPhi^*/\gamma$ and [3, Proposition 15.2] imply that

$$(\forall A \in \mathscr{S}(\mathcal{G}))\quad \big(\mathscr{Q}_A^* + \gamma\varPhi\big)^* = \mathscr{Q}_A \,\square\, (\varPhi^*/\gamma)\colon \mathcal{G} \to \left]-\infty, +\infty\right]\colon z \mapsto \inf_{y\in\mathcal{G}}\Big(\mathscr{Q}_A(y) + \frac{1}{\gamma}\varPhi^*(z-y)\Big). \tag{3.6}$$

Thus, by virtue of (ii), Lemma 2.2(iii), and (3.6),

$$\begin{aligned} (\forall A \in \mathscr{S}(\mathcal{G}))(\forall x \in \mathcal{H})\quad \mathscr{Q}_{L \overset{\gamma}{\blacklozenge} A}(x) &= \big(L \overset{\gamma}{\blacklozenge} \mathscr{Q}_A\big)(x) \\ &= \big(\mathscr{Q}_A^* + \gamma\varPhi\big)^*(Lx) \\ &= \inf_{y\in\mathcal{G}}\underbrace{\Big(\mathscr{Q}_A(y) + \frac{1}{\gamma}\varPhi^*(Lx-y)\Big)}_{\text{affine in } A}. \end{aligned} \tag{3.7}$$

Hence, for every $x \in \mathcal{H}$, the function $\mathscr{S}(\mathcal{G}) \to \mathbb{R}\colon A \mapsto \mathscr{Q}_{L \overset{\gamma}{\blacklozenge} A}(x)$ is concave, as it can be expressed as the infimum of affine functions.

(iv)(a): It follows from (3.3) and Lemma 2.3(iii) that $L \overset{\gamma}{\blacklozenge} B \in \mathscr{S}(\mathcal{H})$. On the other hand, by (1.4) and applying the previous reasoning to B^{-1}, we obtain $L \overset{\gamma}{\diamond} B = (L \overset{1/\gamma}{\blacklozenge} B^{-1})^{-1} \in \mathscr{S}(\mathcal{H})$.

(iv)(b): By Lemma 2.2(i), Lemma 2.1(i), (ii), and (1.4),

$$L \overset{\gamma}{\diamond} \mathscr{Q}_B = \big(L \overset{1/\gamma}{\blacklozenge} \mathscr{Q}_B^*\big)^* = \big(L \overset{1/\gamma}{\blacklozenge} \mathscr{Q}_{B^{-1}}\big)^* = \mathscr{Q}^*_{L \overset{1/\gamma}{\blacklozenge} B^{-1}} = \mathscr{Q}_{(L \overset{1/\gamma}{\blacklozenge} B^{-1})^{-1}} = \mathscr{Q}_{L \overset{\gamma}{\diamond} B}. \tag{3.8}$$

(iv)(c): It follows from Lemma 2.2(iv) and (iv)(b) that

$$(\forall A \in \mathscr{S}(\mathcal{G}))(\forall x \in \mathcal{G})\quad \mathscr{Q}_{L \overset{\gamma}{\diamond} A}(x) = \big(L \overset{\gamma}{\diamond} \mathscr{Q}_A\big)(x) = \inf_{\substack{y\in\mathcal{G}\\ L^*y=x}} \underbrace{\Big(\mathscr{Q}_A(y) + \frac{1}{\gamma}\varPhi(y)\Big)}_{\text{affine in } A} \tag{3.9}$$

Thus, for every $x \in \mathcal{H}$, the function $\mathscr{S}(\mathcal{G}) \to \mathbb{R}\colon A \mapsto \mathscr{Q}_{L \overset{\gamma}{\diamond} A}(x)$ is concave. As a consequence, as in the proof of (iii), R_γ is concave. ☐

The following example shows that, in the finite-dimensional setting, the resolvent composition admits a variational characterization. In particular, this holds for the resolvent average, as established in [4, Proposition 2.8].

Example 3.2 (variational characterization). Suppose that $\mathcal{H}$ and $\mathcal{G}$ are finite-dimensional and that $L \in \mathcal{B}\,(\mathcal{H},\mathcal{G})$ is injective and satisfies $\|L\| \leqslant 1$, let $B \in \mathcal{S}(\mathcal{G})$, and let $\gamma \in \left]0,+\infty\right[$. Define

$$f\colon \mathcal{S}(\mathcal{H}) \to \mathbb{R}\colon X \mapsto -\ln\det\big(X+\gamma^{-1}\mathrm{Id}_{\mathcal{H}}\big) \tag{3.10}$$

and

$$F\colon \mathcal{S}(\mathcal{H}) \to \mathbb{R}\colon X \mapsto f(X) + \big\langle L^{*}\circ(B+\gamma^{-1}\mathrm{Id}_{\mathcal{G}})^{-1}\circ L \,\big|\, X\big\rangle, \tag{3.11}$$

where $\det(X)$ denotes the determinant of X and $\langle X \mid B\rangle$ denotes the trace of $X \circ B$. Then $L \overset{\gamma}{\diamond} B$ is the unique minimizer of F.

Proof. Let $\mathbb{S}(\mathcal{H})$ denote the set of self-adjoint operators on $\mathcal{H}$. Since $\mathbb{S}(\mathcal{H}) \to \mathbb{S}(\mathcal{H})\colon X \mapsto X+\gamma^{-1}\mathrm{Id}_{\mathcal{H}}$ is affine, [3, Example 24.66 and Proposition 8.20] show that f is convex, differentiable, and that $(\forall X \in \mathcal{S}(\mathcal{H}))$ $\nabla f(X) = -(X+\gamma^{-1}\mathrm{Id}_{\mathcal{H}})^{-1}$. Thus, F is also convex and differentiable, being the sum of f and an affine function. Therefore, by virtue of [3, Theorem 16.3 and Proposition 17.31(i)], it suffices to find the critical points of F, that is, to solve $\nabla F(X) = 0$. Altogether, Proposition 3.1(iv)(a) ensures that $L \overset{\gamma}{\diamond} B \in \mathcal{S}(\mathcal{H})$, and

$$\begin{aligned}
\nabla F(X) = 0 &\Leftrightarrow -\big(X+\gamma^{-1}\mathrm{Id}_{\mathcal{H}}\big)^{-1} + L^{*}\circ\big(B+\gamma^{-1}\mathrm{Id}_{\mathcal{G}}\big)^{-1}\circ L = 0\\
&\Leftrightarrow X+\gamma^{-1}\mathrm{Id}_{\mathcal{H}} = L^{*} \triangleright \big(B+\gamma^{-1}\mathrm{Id}_{\mathcal{G}}\big)\\
&\Leftrightarrow X = L \overset{\gamma}{\diamond} B,
\end{aligned} \tag{3.12}$$

which completes the proof. ☐

We now focus on Löwner partial ordering relations for resolvent compositions. These ordering relations will assist us in studying the convergence properties of resolvent compositions $L \overset{\gamma}{\diamond} B$ and $L \overset{\gamma}{\blacklozenge} B$, as well as of the new interpolation $\mathcal{L}_{\gamma}(L,B)$ introduced in Section 5, as γ varies.

Proposition 3.3. *Suppose that* $L \in \mathcal{B}\,(\mathcal{H},\mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $B \in \mathcal{S}(\mathcal{G})$, *and let* $\gamma \in \left]0,+\infty\right[$. *Then the following hold:*

(i) *Set* $\theta = 1/(1+\gamma\|B\|)$. *Then* $\theta(L^{*}\circ B\circ L) \preccurlyeq L \overset{\gamma}{\blacklozenge} B \preccurlyeq L^{*}\circ B\circ L$.

(ii) *Suppose that* $A \in \mathcal{S}(\mathcal{G})$ *satisfies* $A \preccurlyeq B$. *Then* $L \overset{\gamma}{\blacklozenge} A \preccurlyeq L \overset{\gamma}{\blacklozenge} B$.

(iii) *Let* $\rho \in \left]0,+\infty\right[$ *be such that* $\rho \leqslant \gamma$. *Then* $L \overset{\gamma}{\blacklozenge} B \preccurlyeq L \overset{\rho}{\blacklozenge} B$.

Proof. Set $\Psi = \mathrm{Id}_{\mathcal{G}} - L\circ L^{*}$ and recall that $L \overset{\gamma}{\blacklozenge} B = L^{*}\circ(B^{-1}+\gamma\Psi)^{-1}\circ L$ by Lemma 2.4.

(i): Note that $B \preccurlyeq \|B\|\,\mathrm{Id}_{\mathcal{G}}$ and that Lemma 2.1(ii) implies that $\mathrm{Id}_{\mathcal{G}} \preccurlyeq \|B\|\,B^{-1}$. Since $0 \preccurlyeq \Psi \preccurlyeq \mathrm{Id}_{\mathcal{G}}$,

$$B^{-1} \preccurlyeq B^{-1}+\gamma\Psi \preccurlyeq B^{-1}+\gamma\mathrm{Id}_{\mathcal{G}} \preccurlyeq (1+\gamma\|B\|)\,B^{-1}, \tag{3.13}$$

and, by virtue of Lemma 2.1(ii),

$$\theta B \preccurlyeq \big(B^{-1}+\gamma\Psi\big)^{-1} \preccurlyeq B. \tag{3.14}$$

Hence, we deduce from (3.14) and Lemma 2.1(iii) that

$$\theta(L^* \circ B \circ L) \preccurlyeq L \overset{\gamma}{\blacklozenge} B \preccurlyeq L^* \circ B \circ L. \tag{3.15}$$

(ii): Since $\Psi \in \mathcal{P}(\mathcal{G})$, $A^{-1} + \gamma\Psi$ and $B^{-1} + \gamma\Psi$ are in $\mathcal{S}(\mathcal{G})$. Further, by Lemma 2.1(ii) and the fact that $A \preccurlyeq B$, $B^{-1} + \gamma\Psi \preccurlyeq A^{-1} + \gamma\Psi$. Thus, $(A^{-1} + \gamma\Psi)^{-1} \preccurlyeq (B^{-1} + \gamma\Psi)^{-1}$. Altogether, we deduce from Lemma 2.1(iii) that

$$L \overset{\gamma}{\blacklozenge} A = L^* \circ \big(A^{-1} + \gamma\Psi\big)^{-1} \circ L \preccurlyeq L^* \circ \big(B^{-1} + \gamma\Psi\big)^{-1} \circ L = L \overset{\gamma}{\blacklozenge} B. \tag{3.16}$$

(iii): Note that $B^{-1} + \gamma\Psi$ and $B^{-1} + \rho\Psi$ are in $\mathcal{S}(\mathcal{G})$ and that $B^{-1} + \rho\Psi \preccurlyeq B^{-1} + \gamma\Psi$. Therefore, Lemma 2.1(ii)-(iii) yields

$$L \overset{\gamma}{\blacklozenge} B = L^* \circ \big(B^{-1} + \gamma\Psi\big)^{-1} \circ L \preccurlyeq L^* \circ \big(B^{-1} + \rho\Psi\big)^{-1} \circ L = L \overset{\rho}{\blacklozenge} B, \tag{3.17}$$

as claimed. ☐

Corollary 3.4. *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ is bounded below and satisfies $\|L\| \leqslant 1$, let $B \in \mathcal{S}(\mathcal{G})$, and let $\gamma \in \left]0, +\infty\right[$. Then the following hold:*

(i) *Set $\omega = 1 + \|B^{-1}\|/\gamma$. Then $L^* \triangleright B \preccurlyeq L \overset{\gamma}{\diamond} B \preccurlyeq \omega(L^* \triangleright B)$.*

(ii) $L \overset{\gamma}{\blacklozenge} B \preccurlyeq L \overset{\gamma}{\diamond} B$.

(iii) *Suppose that $A \in \mathcal{S}(\mathcal{G})$ satisfies $A \preccurlyeq B$. Then $L \overset{\gamma}{\diamond} A \preccurlyeq L \overset{\gamma}{\diamond} B$.*

(iv) *Let $\rho \in \left]0, +\infty\right[$ be such that $\rho \leqslant \gamma$. Then $L \overset{\gamma}{\diamond} B \preccurlyeq L \overset{\rho}{\diamond} B$.*

Proof. By Proposition 3.1(iv)(a), $L \overset{\gamma}{\diamond} B \in \mathcal{S}(\mathcal{H})$. Further, recall that (1.4) yields $L \overset{\gamma}{\diamond} B = (L \overset{1/\gamma}{\blacklozenge} B^{-1})^{-1}$.

(i): This follows from Lemma 2.1(ii) and Proposition 3.3(i) applied to B^{-1} and $1/\gamma$.

(ii): By Proposition 3.1(ii), Lemma 2.2(ii), and Proposition 3.1(iv)(b),

$$\mathcal{Q}_{L \overset{\gamma}{\blacklozenge} B} = L \overset{\gamma}{\blacklozenge} \mathcal{Q}_B \leqslant L \overset{\gamma}{\diamond} \mathcal{Q}_B = \mathcal{Q}_{L \overset{\gamma}{\diamond} B}. \tag{3.18}$$

Therefore, $L \overset{\gamma}{\blacklozenge} B \preccurlyeq L \overset{\gamma}{\diamond} B$.

(iii): This follows from Lemma 2.1(ii) and Proposition 3.3(ii) applied to B^{-1} and $1/\gamma$.

(iv): This follows from Lemma 2.1(ii) and Proposition 3.3(iii) applied to B^{-1} and $1/\gamma$. ☐

Corollary 3.5. *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ is bounded below and satisfies $\|L\| \leqslant 1$, let $B \in \mathcal{S}(\mathcal{G})$, and set $\kappa = \|B\|\,\|B^{-1}\|$ and $\rho = (1 + \sqrt{\kappa})^2$. Then $L^* \circ B \circ L \preccurlyeq \rho(L^* \triangleright B)$.*

Proof. Set $f \colon \left]0, +\infty\right[\to \left]0, +\infty\right[\colon \gamma \mapsto (1 + \gamma\|B\|)(1 + \|B^{-1}\|/\gamma)$. By Proposition 3.3(i), Corollary 3.4(ii), and Corollary 3.4(i),

$$(\forall \gamma \in \left]0, +\infty\right[) \quad L^* \circ B \circ L \preccurlyeq f(\gamma)\big(L^* \triangleright B\big). \tag{3.19}$$

Since $\rho = \min_{\gamma \in \left]0, +\infty\right[} f(\gamma)$, the assertion follows from (3.19). ☐

We now present the main result of this section. In contrast to [12, Propositions 5.8 and 5.12(i)], which establish graph convergence of resolvent compositions, the following theorem provides asymptotic behavior of resolvent compositions in operator norm, which is stronger than graph convergence and therefore offers additional stability properties.

Theorem 3.6. *Suppose that* $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *and let* $B \in \mathcal{S}(\mathcal{G})$. *Then the following hold:*

(i) $L \overset{\gamma}{\blacklozenge} B \to L^* \circ B \circ L$ *as* $0 < \gamma \to 0$.

(ii) *Suppose that* L *is bounded below. Then* $L \overset{\gamma}{\diamond} B \to L^* \triangleright B$ *as* $\gamma \to +\infty$.

Proof. (i): Set $(\forall \gamma \in \left]0, +\infty\right[)$ $\theta_\gamma = 1/(1 + \gamma\|B\|)$ and $D_\gamma = (L^* \circ B \circ L) - (L \overset{\gamma}{\blacklozenge} B)$. By Proposition 3.3(i),

$$0 \preccurlyeq D_\gamma \preccurlyeq \left(\frac{1 - \theta_\gamma}{\theta_\gamma}\right) (L^* \circ B \circ L). \tag{3.20}$$

In addition, note that $\theta_\gamma \to 1$ as $0 < \gamma \to 0$. Therefore, it follows from (3.20) and Lemma 2.1(iv) that

$$\|D_\gamma\| \leqslant \left(\frac{1 - \theta_\gamma}{\theta_\gamma}\right) \|L^* \circ B \circ L\| \to 0 \quad \text{as} \quad 0 < \gamma \to 0. \tag{3.21}$$

(ii): Set $(\forall \gamma \in \left]0, +\infty\right[)$ $\omega_\gamma = 1 + \|B^{-1}\|/\gamma$ and $D_\gamma = (L \overset{\gamma}{\diamond} B) - (L^* \triangleright B)$. By Corollary 3.4(i),

$$0 \preccurlyeq D_\gamma \preccurlyeq (\omega_\gamma - 1)\ (L^* \triangleright B). \tag{3.22}$$

Also, note that $\omega_\gamma \to 1$ as $\gamma \to +\infty$. Therefore, we combine (3.22) and Lemma 2.1(iv) to obtain

$$\|D_\gamma\| \leqslant (\omega_\gamma - 1)\ \|L^* \triangleright B\| \to 0 \quad \text{as} \quad 0 < \gamma \to +\infty, \tag{3.23}$$

which completes the proof. ☐

Corollary 3.7. *Suppose that* $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ *is bounded below and satisfies* $\|L\| \leqslant 1$. *Then the operator* $R \colon \mathcal{S}(\mathcal{G}) \to \mathcal{S}(\mathcal{H}) \colon A \mapsto L^* \triangleright A$ *is concave in the sense that*

$$(\forall \lambda \in \left]0, 1\right[)(\forall A \in \mathcal{S}(\mathcal{G}))(\forall B \in \mathcal{S}(\mathcal{G})) \quad \lambda(L^* \triangleright A) + (1 - \lambda)(L^* \triangleright B) \preccurlyeq L^* \triangleright \big(\lambda A + (1 - \lambda)B\big). \tag{3.24}$$

Proof. By Proposition 3.1(iv)(c), $R_\gamma \colon \mathcal{S}(\mathcal{G}) \to \mathcal{S}(\mathcal{H}) \colon A \mapsto L \overset{\gamma}{\diamond} A$ is concave, i.e.,

$$(\forall \lambda \in \left]0, 1\right[)(\forall A \in \mathcal{S}(\mathcal{G}))(\forall B \in \mathcal{S}(\mathcal{G})) \quad \lambda(L \overset{\gamma}{\diamond} A) + (1 - \lambda)(L \overset{\gamma}{\diamond} B) \preccurlyeq L^* \overset{\gamma}{\diamond} \big(\lambda A + (1 - \lambda)B\big). \tag{3.25}$$

Hence, letting $\gamma \to +\infty$ in (3.25) and invoking Theorem 3.6(ii) together with Lemma 2.1(v), we obtain (3.24). ☐

Remark 3.8. In the context of the resolvent averages of Example 1.2, Theorem 3.6 and Corollary 3.7 generalize [4, Theorem 4.2 and Corollary 4.6], which were established in the finite-dimensional context using different techniques.

Corollary 3.9. *Suppose that* $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ *is an isometry, and let* $B \in \mathcal{S}(\mathcal{G})$. *Then the following hold:*

(i) $(\forall \gamma \in \left]0, +\infty\right[)$ $L^* \triangleright B \preccurlyeq L \overset{\gamma}{\blacklozenge} B \preccurlyeq L^* \circ B \circ L$.

(ii) $L \overset{\gamma}{\blacklozenge} B \to L^* \circ B \circ L$ *as* $0 < \gamma \to 0$.

(iii) $L \overset{\gamma}{\blacklozenge} B \to L^* \triangleright B$ *as* $\gamma \to +\infty$.

Proof. Since L is an isometry, Lemma 2.5 yields $L \overset{\gamma}{\diamond} B = L \overset{\gamma}{\blacklozenge} B$.

(i): This follows from Proposition 3.3(i) and Corollary 3.4(i).

(ii): This follows from Theorem 3.6(i).

(iii): This follows from Theorem 3.6(ii). $\square$

Corollary 3.10 (resolvent mixtures). *Consider the setting of Example 1.1. Then the following hold:*

(i) $\overset{\blacklozenge}{\mathsf{M}}_\gamma(L_k, B_k)_{1\leqslant k\leqslant p} \preccurlyeq \sum_{k=1}^p \alpha_k L_k^* \circ B_k \circ L_k$.

(ii) $\overset{\blacklozenge}{\mathsf{M}}_\gamma(L_k, B_k)_{1\leqslant k\leqslant p} \to \sum_{k=1}^p \alpha_k L_k^* \circ B_k \circ L_k$ *as* $0 < \gamma \to 0$.

(iii) *Suppose that* L_j *is bounded below for some* $j \in \{1, \ldots, p\}$. *Then the following are satisfied:*

  (a) $\overset{\diamond}{\mathsf{M}}_\gamma(L_k, B_k)_{1\leqslant k\leqslant p} \in \mathcal{S}(\mathcal{H})$ *and* $\overset{\blacklozenge}{\mathsf{M}}_\gamma(L_k, B_k)_{1\leqslant k\leqslant p} \in \mathcal{S}(\mathcal{H})$.

  (b) $\big(\sum_{k=1}^p \alpha_k L_k^* \circ B_k^{-1} \circ L_k\big)^{-1} \preccurlyeq \overset{\diamond}{\mathsf{M}}_\gamma(L_k, B_k)_{1\leqslant k\leqslant p}$.

  (c) $\overset{\diamond}{\mathsf{M}}_\gamma(L_k, B_k)_{1\leqslant k\leqslant p} \to \big(\sum_{k=1}^p \alpha_k L_k^* \circ B_k^{-1} \circ L_k\big)^{-1}$ *as* $\gamma \to +\infty$.

Proof. Note that $L^* \circ B \circ L = \sum_{k=1}^p \alpha_l L_k^* \circ B_k \circ L_k$ and $L^* \triangleright B = \big(\sum_{k=1}^p \alpha_k L_k^* \circ B_k^{-1} \circ L_k\big)^{-1}$. Further, if L_j is bounded below for some $j \in \{1, \ldots, p\}$, then L is also bounded below. Indeed, there exists $\alpha \in\,]0, +\infty[$ such that $(\forall x \in \mathcal{H})\ \alpha\|x\|_{\mathcal{H}} \leqslant \|L_j x\|_{\mathcal{G}_j}$. Thus, L is bounded below since

$$(\forall x \in \mathcal{H}) \quad \|Lx\|_{\mathcal{G}} = \left(\sum_{k=1}^p \alpha_k \|L_k x\|_{\mathcal{G}_k}^2\right)^{1/2} \geqslant \left(\alpha_j \alpha^2 \|x\|_{\mathcal{H}}^2\right)^{1/2} = \big(\alpha_j^{1/2}\alpha\big)\|x\|_{\mathcal{H}}. \tag{3.26}$$

(i): This follows from Proposition 3.3(i).

(ii): This follows from Theorem 3.6(i).

(iii)(a): This follows from Proposition 3.1(iv)(a).

(iii)(b): This follows from Corollary 3.4(i).

(iii)(c): This follows from Theorem 3.6(ii). $\square$

We conclude this section by deriving a result that recovers and extends [4, Theorem 4.2], which was proved in the finite-dimensional setting.

Corollary 3.11. *Consider the setting of Example 1.2. Then the following hold:*

(i) $\big(\sum_{k=1}^p \alpha_k B_k^{-1}\big)^{-1} \preccurlyeq \mathrm{rav}_\gamma(B_k)_{1\leqslant k\leqslant p} \preccurlyeq \sum_{k=1}^p \alpha_k B_k$.

(ii) $\mathrm{rav}_\gamma(B_k)_{1\leqslant k\leqslant p} \to \sum_{k=1}^p \alpha_k B_k$ *as* $0 < \gamma \to 0$.

(iii) $\mathrm{rav}_\gamma(B_k)_{1\leqslant k\leqslant p} \to \big(\sum_{k=1}^p \alpha_k B_k^{-1}\big)^{-1}$ *as* $\gamma \to +\infty$.

Proof. Recall that $\mathrm{rav}_\gamma(B_k)_{1\leqslant k\leqslant p} = \overset{\diamond}{\mathsf{M}}_\gamma(\mathrm{Id}_{\mathcal{H}}, B_k)_{1\leqslant k\leqslant p} = \overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathrm{Id}_{\mathcal{H}}, B_k)_{1\leqslant k\leqslant p}$.

(i): This follows from items (i) and (iii)(b) in Corollary 3.10.

(ii): This follows from Corollary 3.10(ii).

(iii): This follows from Corollary 3.10(iii)(c). $\square$

§4. Nonexpansiveness of resolvent compositions

In this section, we build on the results of Section 3 to prove that the resolvent composition operations are nonexpansive with respect to the Thompson metric [30] on $\mathcal{S}(\mathcal{H})$, defined by

$$\big(\forall A \in \mathcal{S}(\mathcal{H})\big)\big(\forall B \in \mathcal{S}(\mathcal{H})\big) \quad d_T^{\mathcal{H}}(A, B) = \ln\big(\max\{g(A, B), g(B, A)\}\big), \tag{4.1}$$

where $g(A, B) = \inf\big\{\lambda \in \left]0, +\infty\right[\mid A \preccurlyeq \lambda B\big\}$.
The Thompson metric was originally defined on cones in Banach spaces [30]. Since $\mathcal{S}(\mathcal{H})$ is contained in the cone of monotone self-adjoint operators on $\mathcal{B}\,(\mathcal{H})$, which is closed and hence complete in the operator norm topology, and since every $A \in \mathcal{S}(\mathcal{H})$ satisfies $\alpha\mathrm{Id}_{\mathcal{H}} \preccurlyeq A \preccurlyeq \|A\|\mathrm{Id}_{\mathcal{H}}$ for some $\alpha \in \left]0, +\infty\right[$, it follows from [30, Lemma 3] that $(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}})$ is a complete metric space. The metric $d_T^{\mathcal{H}}$ provides a geometric structure on $\mathcal{S}(\mathcal{H})$ that plays a central role in the study of nonlinear matrix equations, especially for establishing existence and uniqueness results via Banach contraction mappings [23, 24, 25, 26], and in various applications to nonlinear optimization [13, 21, 27]. In this context, the nonexpansiveness of resolvent compositions is crucial, as it ensures that the resulting operations preserve both the metric structure and the stability necessary for analysis. For instance, in Section 5, we present two nonlinear equations based on resolvent compositions that admit unique solutions.

Theorem 4.1. *Suppose that $L \in \mathcal{B}\,(\mathcal{H}, \mathcal{G})$ is bounded below and satisfies $\|L\| \leqslant 1$, and let $\gamma \in \left]0, +\infty\right[$. Then the following hold:*

(i) $T_\gamma \colon \big(\mathcal{S}(\mathcal{G}), d_T^{\mathcal{G}}\big) \to \big(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}}\big) \colon B \mapsto L \overset{\gamma}{\blacklozenge} B$ *is nonexpansive, i.e.,*

$$\big(\forall A \in \mathcal{S}(\mathcal{G})\big)\big(\forall B \in \mathcal{S}(\mathcal{G})\big) \quad d_T^{\mathcal{H}}\big(L \overset{\gamma}{\blacklozenge} A, L \overset{\gamma}{\blacklozenge} B\big) \leqslant d_T^{\mathcal{G}}(A, B). \tag{4.2}$$

(ii) $R_\gamma \colon \big(\mathcal{S}(\mathcal{G}), d_T^{\mathcal{G}}\big) \to \big(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}}\big) \colon B \mapsto L \overset{\gamma}{\lozenge} B$ *is nonexpansive, i.e.,*

$$\big(\forall A \in \mathcal{S}(\mathcal{G})\big)\big(\forall B \in \mathcal{S}(\mathcal{G})\big) \quad d_T^{\mathcal{H}}\big(L \overset{\gamma}{\lozenge} A, L \overset{\gamma}{\lozenge} B\big) \leqslant d_T^{\mathcal{G}}(A, B). \tag{4.3}$$

Proof. Let A and B be in $\mathcal{S}(\mathcal{G})$, and set $g(A, B) = \inf\big\{\lambda \in \left]0, +\infty\right[\mid A \preccurlyeq \lambda B\big\}$.
(i): Note that the operator T_γ is well defined by Proposition 3.1(iv)(a). By virtue of (4.1),

$$A \preccurlyeq e^{d_T^{\mathcal{G}}(A,B)} B. \tag{4.4}$$

On the other hand, it follows from [12, Proposition 3.1(vi)] and Proposition 3.3(iii) that

$$(\forall \rho \in [1, +\infty[) \quad L \overset{\gamma}{\blacklozenge} (\rho B) = \rho\big(L \overset{\gamma\rho}{\blacklozenge} B\big) \preccurlyeq \rho\big(L \overset{\gamma}{\blacklozenge} B\big). \tag{4.5}$$

Since $e^{d_T^{\mathcal{G}}(A,B)} \geqslant 1$, we combine Proposition 3.3(ii), (4.4), and (4.5) to obtain

$$L \overset{\gamma}{\blacklozenge} A \preccurlyeq L \overset{\gamma}{\blacklozenge} \big(e^{d_T^{\mathcal{G}}(A,B)} B\big) \preccurlyeq e^{d_T^{\mathcal{G}}(A,B)}\big(L \overset{\gamma}{\blacklozenge} B\big). \tag{4.6}$$

In turn,

$$g\big(L \overset{\gamma}{\blacklozenge} A, L \overset{\gamma}{\blacklozenge} B\big) = \inf\big\{\lambda \in \left]0, +\infty\right[\mid L \overset{\gamma}{\blacklozenge} A \preccurlyeq \lambda(L \overset{\gamma}{\blacklozenge} B)\big\} \leqslant e^{d_T^{\mathcal{G}}(A,B)}. \tag{4.7}$$

By the same argument,

$$g\big(L \overset{\gamma}{\blacklozenge} B, L \overset{\gamma}{\blacklozenge} A\big) \leqslant e^{d_T^{\mathcal{G}}(A,B)}. \tag{4.8}$$

Altogether, it follows from (4.1), (4.7), and (4.8) that

$$d_T^{\mathcal{H}}\big(L \overset{\gamma}{\blacklozenge} A, L \overset{\gamma}{\blacklozenge} B\big) = \max\big\{\ln g\big(L \overset{\gamma}{\blacklozenge} A, L \overset{\gamma}{\blacklozenge} B\big)\,,\, \ln g\big(L \overset{\gamma}{\blacklozenge} B, L \overset{\gamma}{\blacklozenge} A\big)\big\} \leqslant d_T^{\mathcal{G}}(A, B). \tag{4.9}$$

(ii): Note that R_γ is well defined by Proposition 3.1(iv)(a). Since $d_T^{\mathcal{G}}(A, B) = d_T^{\mathcal{G}}(A^{-1}, B^{-1})$, we deduce from (i) and (1.4) that

$$d_T^{\mathcal{H}}\big(L \overset{\gamma}{\diamond} A, L \overset{\gamma}{\diamond} B\big) = d_T^{\mathcal{H}}\big(L \overset{1/\gamma}{\blacklozenge} A^{-1}, L \overset{1/\gamma}{\blacklozenge} B^{-1}\big) \leqslant d_T^{\mathcal{G}}(A^{-1}, B^{-1}) = d_T^{\mathcal{G}}(A, B), \tag{4.10}$$

as announced. ☐

Corollary 4.2. *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ is bounded below and satisfies $\|L\| \leqslant 1$, and let $\gamma \in \left]0, +\infty\right[$. Then the following hold:*

(i) $T_0 \colon \big(\mathcal{S}(\mathcal{G}), d_T^{\mathcal{G}}\big) \to \big(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}}\big) \colon B \mapsto L^* \circ B \circ L$ *is nonexpansive.*

(ii) $R_{+\infty} \colon \big(\mathcal{S}(\mathcal{G}), d_T^{\mathcal{G}}\big) \to \big(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}}\big) \colon B \mapsto L^* \triangleright B$ *is nonexpansive.*

Proof. (i): This follows from Theorem 4.1(i) and Theorem 3.6(i).

(ii): This follows from Theorem 4.1(ii) and Theorem 3.6(ii). ☐

Corollary 4.3. *Consider the setting of Example 1.1. Suppose that L_j is bounded below for some $j \in \{1, \ldots, p\}$ and that, for every $k \in \{1, \ldots, p\}$, $A_k \in \mathcal{S}(\mathcal{G}_k)$, and set $A \colon \mathcal{G} \to \mathcal{G} \colon (y_k)_{1\leqslant k\leqslant p} \mapsto (A_k y_k)_{1\leqslant k\leqslant p}$. Then*

$$d_T^{\mathcal{H}}\Big(\overset{\diamond}{\mathrm{M}}_\gamma(L_k, A_k)_{1\leqslant k\leqslant p}, \overset{\diamond}{\mathrm{M}}_\gamma(L_k, B_k)_{1\leqslant k\leqslant p}\Big) \leqslant d_T^{\mathcal{G}}(A, B) = \max_{1\leqslant k\leqslant p} d_{\mathcal{G}_k}(A_k, B_k) \tag{4.11}$$

and

$$d_T^{\mathcal{H}}\Big(\overset{\blacklozenge}{\mathrm{M}}_\gamma(L_k, A_k)_{1\leqslant k\leqslant p}, \overset{\blacklozenge}{\mathrm{M}}_\gamma(L_k, B_k)_{1\leqslant k\leqslant p}\Big) \leqslant d_T^{\mathcal{G}}(A, B) = \max_{1\leqslant k\leqslant p} d_{\mathcal{G}_k}(A_k, B_k). \tag{4.12}$$

In other words, the resolvent mixtures are nonexpansive for the Thompson metric.

Proof. It is straightforward to verify that $d_T^{\mathcal{G}}(A, B) = \max_{1\leqslant k\leqslant p} d_{\mathcal{G}_k}(A_k, B_k)$. On the other hand, $L \overset{\gamma}{\diamond} A = \overset{\diamond}{\mathrm{M}}_\gamma(L_k, A_k)_{1\leqslant k\leqslant p}$ and $L \overset{\gamma}{\blacklozenge} A = \overset{\blacklozenge}{\mathrm{M}}_\gamma(L_k, A_k)_{1\leqslant k\leqslant p}$. Hence, the assertion follows from Theorem 4.1. ☐

Corollary 4.4 ([19, Theorem 3.5]). *Consider the setting of Example 1.2. Suppose that, for every $k \in \{1, \ldots, p\}$, $A_k \in \mathcal{S}(\mathcal{H})$, and set $A \colon \mathcal{G} \to \mathcal{G} \colon (y_k)_{1\leqslant k\leqslant p} \mapsto (A_k y_k)_{1\leqslant k\leqslant p}$. Then*

$$d_T^{\mathcal{H}}\big(\mathrm{rav}_\gamma(A_k)_{1\leqslant k\leqslant p}, \mathrm{rav}_\gamma(B_k)_{1\leqslant k\leqslant p}\big) \leqslant d_T^{\mathcal{G}}(A, B). \tag{4.13}$$

In other words, the resolvent average is nonexpansive for the Thompson metric.

Proof. Since $\mathrm{rav}_\gamma(A_k)_{1\leqslant k\leqslant p} = \overset{\blacklozenge}{\mathrm{M}}_\gamma(\mathrm{Id}_{\mathcal{H}}, A_k)_{1\leqslant k\leqslant p}$, the conclusion follows from Corollary 4.3. ☐

§5. Geometric means and nonlinear equations

Let $A \in \mathcal{S}(\mathcal{G})$. Since A is strongly positive, there exists $\alpha \in \,]0,+\infty[$ such that $\alpha\mathrm{Id}_{\mathcal{G}} \preccurlyeq A$. Consequently, the spectrum $\sigma(A)$ is contained in the compact interval $[\alpha, \|A\|]$ (see [28, Theorem VI.6 and Problem VII.12]). Hence, for every $t \in \mathbb{R}$, the function $f_t \colon \sigma(A) \to \mathbb{R} \colon \lambda \to \lambda^t$ is well defined and continuous, and the operator $f_t(A) = A^t$ is therefore defined according to the continuous functional calculus (see [28, Section VII]).

Given $A \in \mathcal{S}(\mathcal{G})$ and $B \in \mathcal{S}(\mathcal{G})$, an important instance of Kubo-Ando's operator means [18] is the *t-weighted geometric mean* [1, 19, 22, 26] of A and B, defined by

$$(\forall t \in [0,1]) \quad A\#_t B = A^{1/2} \circ \left(A^{-1/2} \circ B \circ A^{-1/2}\right)^t \circ A^{1/2}. \tag{5.1}$$

From a geometric viewpoint, the curve $t \mapsto A\#_t B$ describes a minimal geodesic between A and B with respect to the Thompson metric (see, e.g., [22, Lemma 2.2(iv)]), in the sense that

$$(\forall t \in [0,1])(\forall s \in [0,1]) \quad d_T^{\mathcal{G}}(A\#_t B, A\#_s B) = |t-s|\, d_T^{\mathcal{G}}(A,B). \tag{5.2}$$

In particular, the geometric mean $A\#B = A\#_{1/2}B$ is the metric midpoint of the arithmetic mean $(A+B)/2$ and the harmonic mean $2(A^{-1}+B^{-1})^{-1}$ for the Thompson metric (see [11, 20]).

The following result introduces a new interpolation between $L^* \triangleright B$ and $L^* \circ B \circ L$, which generalizes the weighted $\mathscr{A}\#\mathscr{H}$–mean discussed in Example 1.3.

Proposition 5.1. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is an isometry, let $B \in \mathcal{S}(\mathcal{G})$, and let $\gamma \in \,]0,+\infty[$. Define*

$$\mathcal{L}_\gamma(L,B) = \big(L^* \circ (B + \gamma\mathrm{Id}_{\mathcal{G}}) \circ L\big)\#\big(L^* \triangleright (B + \gamma\mathrm{Id}_{\mathcal{G}})\big) - \gamma\mathrm{Id}_{\mathcal{H}} \tag{5.3}$$

and

$$\mathcal{L}_{-\gamma}(L,B) = \left(\mathcal{L}_\gamma(L,B^{-1})\right)^{-1}. \tag{5.4}$$

Then the following hold:

(i) $L^* \triangleright B \preccurlyeq \mathcal{L}_{-\gamma}(L,B) \preccurlyeq L \overset{\gamma}{\blacklozenge} B \preccurlyeq \mathcal{L}_{1/\gamma}(L,B) \preccurlyeq L^* \circ B \circ L$.

(ii) $\mathcal{L}_\gamma(L,B) \to L^* \circ B \circ L$ *as* $\gamma \to +\infty$.

(iii) $\mathcal{L}_\gamma(L,B) \to L^* \triangleright B$ *as* $\gamma \to -\infty$.

Proof. (i): Since L is an isometry, $L^* \circ L = \mathrm{Id}_{\mathcal{H}}$ and Lemma 2.5 yields $L \overset{\gamma}{\diamond} B = L \overset{\gamma}{\blacklozenge} B$. By Corollary 3.9(i), (5.3), and the fact that $B\#B = B$,

$$\begin{aligned}
\mathcal{L}_{1/\gamma}(L,B) &\preccurlyeq \big(L^* \circ (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}) \circ L\big)\#\big(L^* \circ (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}) \circ L\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}}\\
&= \big(L^* \circ (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}) \circ L\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}}\\
&= L^* \circ B \circ L + \gamma^{-1}(L^* \circ L - \mathrm{Id}_{\mathcal{H}})\\
&= L^* \circ B \circ L.
\end{aligned} \tag{5.5}$$

Similarly, (1.3), Corollary 3.9(i), and (5.3), imply that

$$\begin{aligned}
L \overset{\gamma}{\diamond} B &= L^* \triangleright (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}}\\
&= \big(L^* \triangleright (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}})\big)\#\big(L^* \triangleright (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}})\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}}\\
&\preccurlyeq \big(L^* \circ (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}}) \circ L\big)\#\big(L^* \triangleright (B + \gamma^{-1}\mathrm{Id}_{\mathcal{G}})\big) - \gamma^{-1}\mathrm{Id}_{\mathcal{H}}\\
&= \mathcal{L}_{1/\gamma}(L,B).
\end{aligned} \tag{5.6}$$

Thus, (5.5) and (5.6) yield

$$L \overset{\gamma}{\blacklozenge} B \preccurlyeq \mathcal{L}_{1/\gamma}(L,B) \preccurlyeq L^* \circ B \circ L. \tag{5.7}$$

On the other hand, by virtue of Lemma 2.1(ii), (5.7) applied to B^{-1} and $1/\gamma$, (5.3), and (1.4),

$$L^* \triangleright B = (L^* \circ B^{-1} \circ L)^{-1} \preccurlyeq \mathcal{L}_{\gamma}(L,B^{-1})^{-1} = \mathcal{L}_{-\gamma}(L,B) \preccurlyeq \big(L \overset{1/\gamma}{\blacklozenge} B^{-1}\big)^{-1} = L \overset{\gamma}{\diamond} B = L \overset{\gamma}{\blacklozenge} B. \tag{5.8}$$

Hence, the result follows from (5.7) and (5.8).

(ii): This follows from (i) and Corollary 3.9(ii).

(iii): This follows from (i) and Corollary 3.9(iii). ☐

Remark 5.2. Note that the operator $\mathcal{L}_\gamma(L,B)$ is a type of weighted geometric mean that interpolates between the parallel composition $L^* \triangleright B$ ($\gamma \to -\infty$) and $L^* \circ B \circ L$ ($\gamma \to +\infty$). In the particular case where L and B are defined as in Example 1.2, $L^* \circ B \circ L = \sum_{k=1}^{p} \alpha_k B_k$ is the arithmetic average, $L^* \triangleright B = \big(\sum_{k=1}^{p} \alpha_k B_k^{-1}\big)^{-1}$ is the harmonic average, and $\mathcal{L}_\gamma(L,B)$ reduces to the *weighted $\mathscr{A}\#\mathscr{H}$-mean* with parameter γ of Example 1.3, with Proposition 5.1(ii)–(iii) recovering [17, Proposition 3.4].

We now focus on nonlinear equations that are based on resolvent compositions. The nonexpansive nature of these operations, as shown in Section 4, will play a key role in our subsequent analysis.

Proposition 5.3. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is bounded below and satisfies $\|L\| \leqslant 1$, let $B \in \mathcal{S}(\mathcal{G})$, let $\gamma \in \left]0,+\infty\right[$, and let $t \in \left]0,1\right[$. Set*

$$\varphi\colon \big(\mathcal{S}(\mathcal{G}), d_T^{\mathcal{G}}\big) \to \big(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}}\big)\colon X \mapsto L \overset{\gamma}{\blacklozenge} (X \#_t B). \tag{5.9}$$

Then the following hold:

(i) *φ is $(1-t)$-Lipschitzian.*

(ii) *Suppose that $\mathcal{H} = \mathcal{G}$. Then the problem*

$$\text{find} \quad X \in \mathcal{S}(\mathcal{H}) \quad \text{such that} \quad X = L \overset{\gamma}{\blacklozenge} (X \#_t B) \tag{5.10}$$

admits a unique solution.

Proof. (i): It follows from Theorem 4.1(i) and [22, Lemma 2.2(iii)] that

$$\begin{aligned}(\forall X \in \mathcal{S}(\mathcal{G}))(\forall Y \in \mathcal{S}(\mathcal{G})) \quad d_T^{\mathcal{H}}\big(\varphi(X),\varphi(Y)\big) &= d_T^{\mathcal{H}}\big(L \overset{\gamma}{\blacklozenge} (X\#_t B), L \overset{\gamma}{\blacklozenge} (Y\#_t B)\big)\\ &\leqslant d_T^{\mathcal{G}}(X\#_t B, Y\#_t B)\\ &\leqslant (1-t)d_T^{\mathcal{G}}(X,Y) + t d_T^{\mathcal{G}}(B,B)\\ &= (1-t)d_T^{\mathcal{G}}(X,Y).\end{aligned} \tag{5.11}$$

(ii): Since $d_T^{\mathcal{H}}$ is a complete metric on $\mathcal{S}(\mathcal{H})$, (i) and the Banach–Picard theorem [3, Theorem 1.50] ensure that φ admits a unique fixed point, i.e., (5.10) admits a unique solution. ☐

Remark 5.4. Let $X \in \mathcal{S}(\mathcal{H})$ be the unique solution to (5.10). Since $(X\#_t B)^{-1} = X^{-1}\#_t B^{-1}$ and $L \overset{\gamma}{\blacklozenge} B = (L \overset{1/\gamma}{\diamond} B^{-1})^{-1}$, we note that X^{-1} is the unique solution to the problem

$$\text{find} \quad Y \in \mathcal{S}(\mathcal{H}) \quad \text{such that} \quad Y = L \overset{1/\gamma}{\diamond} (Y\#_t B^{-1}). \tag{5.12}$$

Proposition 5.5. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is bounded below and satisfies $\|L\| \leqslant 1$, let $B \in \mathcal{S}(\mathcal{G})$, let $\gamma \in \left]0,+\infty\right[$, let $t \in \left]-1,1\right[$, and set*

$$\varphi\colon \big(\mathcal{S}(\mathcal{G}), d_T^{\mathcal{G}}\big) \to \big(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}}\big) \colon X \mapsto L \overset{\gamma}{\blacklozenge} (B^* \circ X^t \circ B). \tag{5.13}$$

Then the following hold:

(i) *φ is $|t|$-Lipschitzian.*

(ii) *Suppose that $\mathcal{H} = \mathcal{G}$. Then the problem*

$$\text{find} \quad X \in \mathcal{S}(\mathcal{H}) \quad \text{such that} \quad X = L \overset{\gamma}{\blacklozenge} (B^* \circ X^t \circ B) \tag{5.14}$$

admits a unique solution.

Proof. Since $B \in \mathcal{S}(\mathcal{G})$, for every $X \in \mathcal{S}(\mathcal{G})$, $B^* \circ X^t \circ B \in \mathcal{S}(\mathcal{G})$. Hence, φ is well defined by virtue of Proposition 3.1(iv)(a).

(i): Let $X \in \mathcal{S}(\mathcal{G})$ and $Y \in \mathcal{S}(\mathcal{G})$. By [22, Lemma 2.2(i)],

$$d_T^{\mathcal{G}}(B^* \circ X^t \circ B, B^* \circ Y^t \circ B) = d_T^{\mathcal{G}}(X^t, Y^t) = d_T^{\mathcal{G}}(X^{|t|}, Y^{|t|}) \tag{5.15}$$

Thus, combining Theorem 4.1(i), (5.15), and [22, Lemma 2.2(iii)],

$$\begin{aligned} d_T^{\mathcal{H}}\big(L \overset{\gamma}{\blacklozenge} (B^* \circ X^t \circ B), L \overset{\gamma}{\blacklozenge} (B^* \circ Y^t \circ B)\big) &\leqslant d_T^{\mathcal{G}}(B^* \circ X^t \circ B, B^* \circ Y^t \circ B) \\ &= d_T^{\mathcal{G}}(X^{|t|}, Y^{|t|}) \\ &= d_T^{\mathcal{G}}(\mathrm{Id}_{\mathcal{G}} \#_{|t|} X, \mathrm{Id}_{\mathcal{G}} \#_{|t|} Y) \\ &\leqslant |t| d_T^{\mathcal{G}}(X, Y). \end{aligned} \tag{5.16}$$

(ii): This follows from (i) and the Banach–Picard theorem. ☐

Corollary 5.6. *Consider the setting of Example 1.1. Suppose that L_j is bounded below for some $j \in \{1,\ldots,p\}$ and that, for every $k \in \{1,\ldots,p\}$, $\mathcal{G}_k = \mathcal{H}$, and let $s \in \left]0,1\right[$ and $t \in \left]-1,1\right[$. Then the problems*

$$\text{find} \quad X \in \mathcal{S}(\mathcal{H}) \quad \text{such that} \quad X = \overset{\blacklozenge}{\mathrm{M}}_{\gamma}\big(L_k, X \#_s B_k\big)_{1 \leqslant k \leqslant p} \tag{5.17}$$

and

$$\text{find} \quad X \in \mathcal{S}(\mathcal{H}) \quad \text{such that} \quad X = \overset{\blacklozenge}{\mathrm{M}}_{\gamma}\big(L_k, B_k^* \circ X^t \circ B_k\big)_{1 \leqslant k \leqslant p} \tag{5.18}$$

admit unique solutions.

Proof. Set $R\colon \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{G})\colon X \mapsto \boldsymbol{X}$, where $\boldsymbol{X}\colon \mathcal{G} \to \mathcal{G}\colon (y_k) \mapsto (X y_k)_{1 \leqslant k \leqslant p}$, and set

$$\varphi_1\colon \big(\mathcal{S}(\mathcal{G}), d_T^{\mathcal{G}}\big) \to \big(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}}\big) \colon X \mapsto L \overset{\gamma}{\blacklozenge} (X \#_s B) \tag{5.19}$$

and

$$\varphi_2\colon \big(\mathcal{S}(\mathcal{G}), d_T^{\mathcal{G}}\big) \to \big(\mathcal{S}(\mathcal{H}), d_T^{\mathcal{H}}\big) \colon X \mapsto L \overset{\gamma}{\blacklozenge} (B^* \circ X^t \circ B). \tag{5.20}$$

Note that $(\forall\lambda\in\,]0,+\infty[)\ X\preccurlyeq\lambda Y\Rightarrow\boldsymbol{\mathcal{X}}\preccurlyeq\lambda\boldsymbol{\mathcal{Y}}$. Thus,

$$g(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}})=\inf\big\{\lambda\in\,]0,+\infty[\;\big|\;\boldsymbol{\mathcal{X}}\preccurlyeq\lambda\boldsymbol{\mathcal{Y}}\big\}\leqslant\inf\big\{\lambda\in\,]0,+\infty[\;\big|\;X\preccurlyeq\lambda Y\big\}=g(X,Y), \tag{5.21}$$

and it follows from (4.1) that

$$d_T^{\mathcal{G}}\big(R(X),R(Y)\big)=d_T^{\mathcal{G}}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}})\leqslant d_T^{\mathcal{H}}(X,Y). \tag{5.22}$$

Now, given that R is nonexpansive, Propositions 5.3(i) implies that $\varphi_1\circ R$ is $(1-s)$-Lipschitzian, whereas Proposition 5.5(i) implies that $\varphi_2\circ R$ is $|t|$-Lipschitzian. Further, since $\boldsymbol{\mathcal{X}}\#_sB\colon\mathcal{G}\to\mathcal{G}\colon(y_k)_{1\leqslant k\leqslant p}\mapsto\big((X\#_sB_k)y_k\big)_{1\leqslant k\leqslant p}$ and $B^*\circ X^t\circ B\colon\mathcal{G}\to\mathcal{G}\colon(y_k)_{1\leqslant k\leqslant p}\mapsto\big((B_k^*\circ X^t\circ B_k)y_k\big)_{1\leqslant k\leqslant p}$, we deduce that

$$\varphi_1\circ R\colon\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})\colon X\mapsto L\overset{\gamma}{\blacklozenge}(\boldsymbol{\mathcal{X}}\#_sB)=\overset{\blacklozenge}{\mathrm{M}}_\gamma(L_k,X\#_sB_k)_{1\leqslant k\leqslant p} \tag{5.23}$$

and

$$\varphi_2\circ R\colon\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})\colon X\mapsto L\overset{\gamma}{\blacklozenge}(B^*\circ X^t\circ B)=\overset{\blacklozenge}{\mathrm{M}}_\gamma(L_k,B_k^*\circ X^t\circ B_k)_{1\leqslant k\leqslant p}. \tag{5.24}$$

Altogether, it follows from the Banach–Picard theorem that $\varphi_1\circ R$ and $\varphi_2\circ R$ admit unique fixed points, i.e., the problems (5.17) and (5.18) admit unique solutions. ☐

Corollary 5.7. *Consider the setting of Corollary 5.6. Then the problems*

$$\textit{find}\quad X\in\mathcal{S}(\mathcal{H})\quad\textit{such that}\quad X=\sum_{k=1}^{p}\alpha_kL_k^*\circ(X\#_sB_k)\circ L_k \tag{5.25}$$

and

$$\textit{find}\quad X\in\mathcal{S}(\mathcal{H})\quad\textit{such that}\quad X=\sum_{k=1}^{p}\alpha_kL_k^*\circ(B_k^*\circ X^t\circ B_k)\circ L_k \tag{5.26}$$

admit unique solutions.

Proof. As shown in the proof of Corollary 5.6, the operators

$$\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})\colon X\mapsto\overset{\blacklozenge}{\mathrm{M}}_\gamma(L_k,X\#_sB_k)_{1\leqslant k\leqslant p} \tag{5.27}$$

and

$$\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})\colon X\mapsto\overset{\blacklozenge}{\mathrm{M}}_\gamma(L_k,B_k^*\circ X^t\circ B_k)_{1\leqslant k\leqslant p} \tag{5.28}$$

are $(1-s)$-Lipschitzian and $|t|$-Lipchitzian, respectively. Therefore, by virtue of Corollary 3.10(ii), letting $\gamma\to0$, we deduce that operators

$$\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})\colon X\mapsto\sum_{k=1}^{p}\alpha_kL_k^*\circ(X\#_sB_k)\circ L_k \tag{5.29}$$

and

$$\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})\colon X\mapsto L_k^*\circ(B_k^*\circ X^t\circ B_k)\circ L_k \tag{5.30}$$

are $(1-s)$-Lipschitzian and $|t|$-Lipchitzian, respectively. Finally, the conclusion follows from the Banach–Picard theorem. ☐

Corollary 5.8 ([19, Theorem 4.2]). *Consider the setting of Example 1.2, and let* $s \in \,]0,1[$ *and* $t \in \,]-1,1[$. *Then the problems*

$$\text{find} \quad X \in \mathcal{S}(\mathcal{H}) \quad \text{such that} \quad X = \operatorname{rav}_\gamma(X\#_s B_k)_{1\leqslant k\leqslant p} \tag{5.31}$$

and

$$\text{find} \quad X \in \mathcal{S}(\mathcal{H}) \quad \text{such that} \quad X = \operatorname{rav}_\gamma(B_k^* \circ X^t \circ B_k)_{1\leqslant k\leqslant p} \tag{5.32}$$

admit unique solutions.

Proof. A direct consequence of Corollary 5.6. ☐

Corollary 5.9 ([22, Theorem 3.1]). *Consider the setting of Example 1.2 and let* $s \in \,]0,1[$. *Then the problem*

$$\text{find} \quad X \in \mathcal{S}(\mathcal{H}) \quad \text{such that} \quad X = \sum_{k=1}^{p} \alpha_k (X\#_s B_k) \tag{5.33}$$

admits a unique solution.

Proof. A direct consequence of Corollary 5.7. ☐

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